The Laplace Transform

Integral transforms are useful tools for solving linear differential equations.

An integral transform of a function $f(t)$ is a function of the form

$$F(s) = \int_{\alpha}^{\beta} K(s,t) f(t)\ dt,$$

where $K(s,t)$ is a given function, called the **kernel** of the transformation. The limits α and β include real numbers as well as ∞ and $-\infty$.

We will focus on a particular type of integral transform called the **Laplace Transform**.

To understand the Laplace transform, we need to know how to compute an **improper integral**:

$$\int_{a}^{\infty} f(t)\ dt = \lim_{u \to \infty} \int_{a}^{u} f(t)\ dt.$$

If this limit exists, then the integral is said to **converge**. If the limit does not exist, then the integral is said to **diverge**, or **fail to exist**.

Example. *Let c be some real nonzero constant. Compute*

$$\int_0^\infty e^{ct}\ dt.$$

This is an improper integral, so we compute the limiting value of the corresponding finite integral:

$$\begin{aligned}
\int_0^\infty e^{ct}\ dt &= \lim_{u\to\infty}\int_0^u e^{ct}\ dt \\
&= \lim_{u\to\infty}\left\{\frac{e^{ct}}{c}\right\}_{t=0}^{t=u} \\
&= \lim_{u\to\infty}\left(\frac{e^{uc}}{c}-\frac{1}{c}\right) \\
&= \begin{cases} -1/c, & \text{if } c<0, \\ \text{The integral diverges}, & \text{if } c\geq 0. \end{cases}
\end{aligned}$$

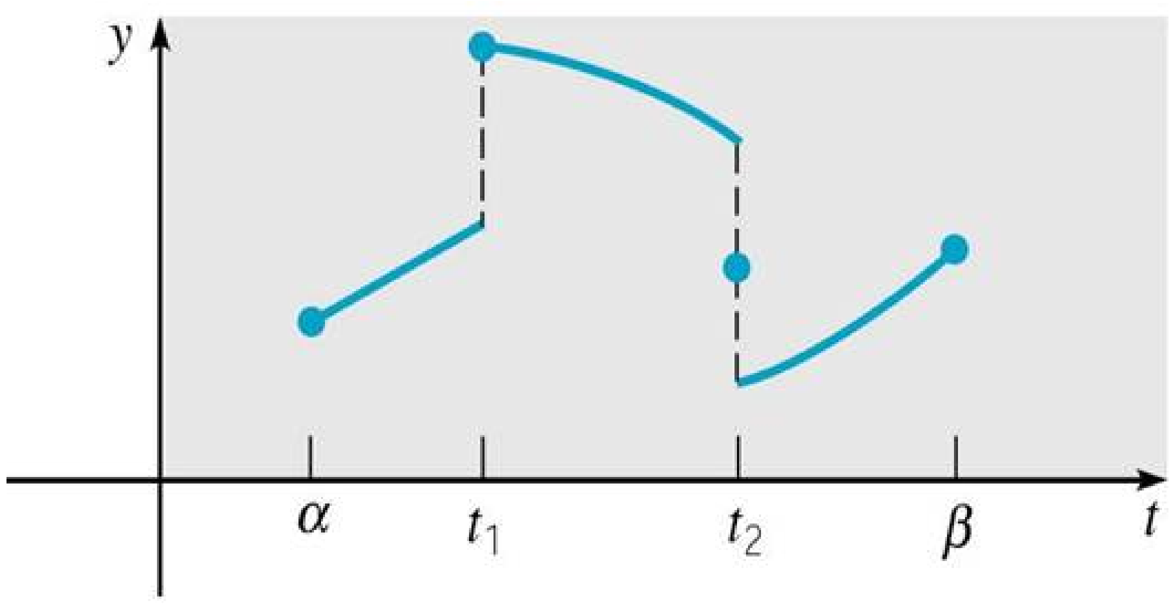
In our later discussion, we will encounter a number of important applications in which we attempt to integrate functions that are not continuous.

A function f is said to be **piecewise continuous** on an interval $[\alpha, \beta]$ if the interval can be partitioned into a finite number of points

$$\alpha = t_0 < t_1 < \cdots < t_n = \beta$$

so that

- f is continuous on each open subinterval $t_{i-1} < t < t_i$.
- f approaches a finite limit as the endpoints of each subinterval are approached from within the subinterval.



In short, a function is piecewise continuous, if it "jumps" at a finite number of points in the interval.

The **Laplace Transform** of a function $f(t)$ is a function $F(s)$, defined by the equation

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\ dt.$$

Notice that, in the definition above,

- s is a free parameter, and
- t is the variable of integration.

Therefore, the resulting integral will be a function of s, if the corresponding limit exists.

The Laplace transform of some functions does not exist.

We need the following theorem to determine when the Laplace transform of a function represents a convergent integral.

Comparison Test for Integrals: Suppose f is a a piecewise continuous function.

- If $|f(t)| \le g(t)$ when $t \ge M$ for some positive constant M, and if $\displaystyle\int_M^\infty g(t)\ dt$ converges, then $\displaystyle\int_a^\infty f(t)\ dt$ also converges.
- If $f(t) \ge g(t) \ge 0$ for $t \ge M$, and if $\displaystyle\int_M^\infty g(t)\ dt$ diverges, then $\displaystyle\int_a^\infty f(t)\ dt$ also diverges.

Example. *Use the comparison test for integrals to determine a sufficient condition on a given piecewise continuous function $f(t)$ so that the Laplace transform of $f(t)$ exists.*

By the comparison test, we need to insist that $f(t)$ is eventually bounded by some "nice" function $g(t)$, so that the integral arising from the Laplace transform of $g(t)$ converges.

Let's take $g(t) = Ke^{at}$, for some constants K and a.

For which such constants does $\mathcal{L}\{g(t)\}$ exist?

The Laplace transform of $g(t)$ is

$$\begin{aligned}
\mathcal{L}\{g(t)\} &= \int_0^\infty Ke^{-st}e^{at}\, dt \\
&= K\int_0^\infty e^{at-st}\, dt \\
&= K\int_0^\infty e^{(a-s)t}\, dt = \begin{cases} -\frac{K}{a-s}, & \text{if } a-s<0, \\ \text{The integral diverges,} & \text{if } a-s \geq 0. \end{cases} \\
&\qquad\qquad\qquad\quad = \begin{cases} -\frac{K}{a-s}, & \text{if } s>a, \\ \text{The integral diverges,} & \text{if } s \leq a. \end{cases}
\end{aligned}$$

By the comparison test for integrals, it suffices to require that $f(t)$ be eventually bounded by $g(t) = Ke^{at}$. Then the Laplace transform of $f(t)$ will exist for $s > a$.

This proves the following theorem.

Theorem. *Suppose that f is a piecewise continuous function on the interval $0 \le t \le A$ for any positive A. Assume that*

$$|f(t)| \le Ke^{at} \qquad \text{when } t \ge M.$$

Then the Laplace transform of $f(t)$ exists for $s > a$.

In a more general context, we allow the parameter s to be complex. In this case, the existence theorem above can be phrased in more general terms.

Example. *Compute the Laplace transform of the constant function $f(t) = 1$, and give the interval of existence.*

The constant function 1 satisfies $|1| \le e^{at}$ for $a \ge 0$. Thus, we require that $s > 0$.

In this case,

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1 \; dt = \lim_{u\to\infty} \left\{ -\frac{e^{-st}}{s} \right\}_{t=0}^{t=u} = \frac{1}{s}.$$

Example. *Show that, if the Laplace transform of f and g exists for $s > a_1$ and $s > a_2$, respectively, then*

$$\mathcal{L}\{c_1 f(t) + c_2 g(t)\} = c_1 \mathcal{L}\{f(t)\} + c_2 \mathcal{L}\{g(t)\},$$

provided that

$$s > \max\{a_1, a_2\}.$$

By applying the definition of the Laplace transform, we see that

$$\mathcal{L}\{c_1 f(t) + c_2 g(t)\} = \int_0^\infty e^{-st}\{c_1 f(t) + c_2 g(t)\} dt$$

$$= c_1 \int_0^\infty e^{-st} f(t)\ dt + c_2 \int_0^\infty e^{-st} g(t)\ dt$$

$$= c_1 \mathcal{L}\{f(t)\} + c_2 \mathcal{L}\{g(t)\}.$$

The last two terms above represent convergent integrals as long as s is greater than $\max\{a_1, a_2\}$.

The above example shows that the Laplace transform $\mathcal{L}$ is a **linear operator**.

Example. *Find the Laplace transform of*

$$f(t) = 5e^{-2t} - 3\sin 4t.$$

Since the Laplace transform is linear, we can compute the transform of each term in $f(t)$ separately.

The Laplace transform of e^{-2t} is convergent for $s > -2$ since $|e^{-2t}| \le e^{at}$, for any $a \ge -2$. Therefore,

$$\mathcal{L}\{e^{-2t}\} = \int_0^\infty e^{-st}e^{-2t}dt = \int_0^\infty e^{(-s-2)t}dt = \frac{1}{s+2}, \qquad s > -2.$$

The laplace transform of $\sin 4t$ is valid for $s > 0$ since $|\sin 4t| \le e^{at}$ for any $a \ge 0$. The transform itself is

$$\begin{aligned}
\mathcal{L}\{\sin 4t\} &= \int_0^\infty e^{-st}\sin 4t \; dt \\
&= \lim_{u\to\infty} \int_0^u e^{-st}\sin 4t \; dt \\
&= \lim_{u\to\infty} \left[\left\{ -\frac{e^{-st}\cos 4t}{4} \right\}_{t=0}^{t=u} - \frac{s}{4}\int_0^u e^{-st}\cos 4t \; dt \right] \\
&= \frac{1}{4} - \frac{s}{4}\int_0^\infty e^{-st}\cos 4t \; dt \\
&= \lim_{u\to\infty} \left[\frac{1}{4} - \frac{s}{4}\left(\left\{ \frac{e^{-st}\sin 4t}{4} \right\}_{t=0}^{t=u} + \frac{s}{4}\int_0^u e^{-st}\sin 4t \; dt \right) \right] \\
&= \frac{1}{4} - \frac{s^2}{16}\int_0^\infty e^{-st}\sin 4t \; dt \\
&= \frac{1}{4} - \frac{s^2}{16}\mathcal{L}\{\sin 4t\}.
\end{aligned}$$

We now use the equality above to solve for $\mathcal{L}\{\sin 4t\}$:

That is,

$$\mathcal{L}\{\sin 4t\} + \frac{s^2}{16}\mathcal{L}\{\sin 4t\} = \frac{1}{4}.$$

implies

$$\mathcal{L}\{\sin 4t\} = \frac{1/4}{1+\frac{s^2}{16}} = \frac{\frac{1}{4}\cdot 16}{s^2+16} = \frac{4}{s^2+16}.$$

Putting together the facts

$$\mathcal{L}\{e^{-2t}\} = \frac{1}{s+2} \qquad \text{and} \qquad \mathcal{L}\{\sin 4t\} = \frac{1/4}{1+\frac{s^2}{16}} = \frac{4}{s^2+16},$$

we see that

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{5e^{-2t} - 3\sin 4t\} \\ &= 5\mathcal{L}\{e^{-2t}\} - 3\mathcal{L}\{\sin 4t\} \\ &= \frac{5}{s+2} - \frac{12}{s^2+16}. \end{aligned}$$

The Laplace transform obtained here is valid for

$$s > \max\{-2, 0\} = 0.$$