The Method of Undetermined Coefficients for Higher Order Differential Equations

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The method of undetermined coefficients works for finding particular solutions to higher order equations.

The key difference is that, if our guess is a multiple of the solution to the corresponding homogeneous equation, then we may need to multiply that guess by

$$t, \qquad t^2, \qquad t^3, \qquad t^4, \qquad \text{and so on,}$$

up the *order* of the differential equation, in order to make things work.

Otherwise, the same procedure applies.

We make a guess for the form of the solution based upon the form of $g(t)$ in

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t).$$

The general solution to a nonhomogeneous equation is found by adding a particular solution of the nonhomogeneous equation to an arbitrary linear combination of the n linear independent solutions to the homogeneous equation:

$$y(t) = \underbrace{Y(t)}_{\text{P.S.}} + \underbrace{c_1y_1(t) + c_2y_2(t) + \cdots c_ny_n(t)}_{\text{Gen. solution to homogeneous}}.$$

Example. *Find a particular solution to the equation*

$$y^{(4)} + 2y'' + y = 3\sin t - 5\cos t.$$

The corresponding homogeneous equation is

$$y^{(4)} + 2y'' + y = 0.$$

We solved this equation previously by showing that the characteristic equation

$$r^4 + 2r^2 + 1 = [(r+i)(r-i)]^2 = 0,$$

so that the general solution to the homogeneous equation is

$$y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t.$$

From the form of $g(t) = 3\sin t - 5\cos t$, we would normally guess that a solution to the nonhomogeneous equation would be of the form

$$Y(t) = A\cos t + B\sin t.$$

However, each term here is a solution to the homogeneous equation.

Our next guess,

$$Y(t) = At\cos t + Bt\sin t$$

also includes solutions to the homogeneous equation.

Therefore, our guess should be

$$Y(t) = At^2\cos t + Bt^2\sin t.$$

Now we need to take all four derivatives of our guess and plug the resulting expressions into the differential equation to find A and B.

After computing these, plugging them into the differential equation, and simplifying, we find that

$$-8A\cos t - 8B\sin t = 3\sin t - 5\cos t,$$

so that

$$A = \frac{5}{8}, \qquad B = -\frac{3}{8}.$$

Example. *Find the solution to the intial value problem*

$$y''' - 3y'' + 2y' = t + e^t,$$

$$y(0) = 1, \qquad y'(0) = -\frac{1}{4}, \qquad y''(0) = -\frac{3}{2}.$$

We first solve the corresponding homogeneous equation

$$y''' - 3y'' + 2y' = 0.$$

The characteristic equation is

$$r^3 - 3r^2 + 2r = r(r^2 - 3r + 2) = r(r-1)(r-2) = 0.$$

Thus, the general solution to the homogeneous equation is

$$y(t) = c_1 + c_2 e^t + c_3 e^{2t}.$$

To find the general solution to the nonhomogeneous equation, need a particular solution to the nonhomogeneous equation. Since $g(t) = t + e^t$ is the sum of two terms, containing functions of different forms, we consider the two equations

$$y''' - 3y'' + 2y' = t, \tag{1}$$

$$y''' - 3y'' + 2y' = e^t. \tag{2}$$

For equation (1), we guess

$$Y(t) = At + B.$$

Then we *think again* and notice that B is a multiple of a solution to the homogeneous equation, so that our guess needs to be multiplied by t. That is, we guess that a solution is of the form

$$Y(t) = At^2 + Bt.$$

When we plug in $Y(t)$, the DE (1) reduces to

$$-6A + 2B + 4At = t,$$

so that

$$-6A + 2B = 0, \qquad \text{and} \qquad 4A = 1.$$

Therefore,

$$A = \frac{1}{4}, \qquad B = \frac{3}{4}.$$

In making the guess for a solution to equation (2), our first impulse is to use $Y(t) = De^t$, but this function satisfies the homogeneous equation. Hence, we use $Y(t) = Dte^t$. For this $Y(t)$, the differential equation reduces to

$$-De^t = e^t \qquad \Longrightarrow \qquad D = -1.$$

So our particular solution to the nonhomogeneous equation is

$$Y(t) = \frac{1}{4}t^2 + \frac{3}{4}t - te^t.$$

Using the general solution to the homogeneous equation, we see that the general solution to the nonhomogeneous equation is

$$y(t) = c_1 + c_2e^t + c_3e^{2t} + \frac{1}{4}t^2 + \frac{3}{4}t - te^t.$$

We need to find the constants c_1, c_2, c_3 such that the initial conditions

$$y(0) = 1, \qquad y'(0) = -\frac{1}{4}, \qquad y''(0) = -\frac{3}{2}$$

are satisfied. We compute

$$y'(t) = \frac{3}{4} - e^t + \frac{t}{2} - te^t + c_2e^t + 2c_3e^{2t}$$

and

$$y''(t) = \frac{1}{2} - 2e^t - te^t + c_2e^t + 4c_3e^{2t}.$$

Thus,

$$\begin{aligned} 1 &= y(0) = c_1 + c_2 + c_3 \\ -\frac{1}{4} &= y'(0) = c_2 + 2c_3 - \frac{1}{4} \\ -\frac{3}{2} &= y''(0) = -\frac{3}{2} + c_2 + 4c_3 \end{aligned}$$

Solving this system, we find that

$$c_1 = 1, \qquad c_2 = 0, \qquad c_3 = 0.$$

Therefore, the solution to the initial value problem is

$$y(t) = 1 + \frac{1}{4}t^2 + \frac{3}{4}t - te^t.$$

Example. *Determine the suitable form of a "guess" to a particular solution to the following differential equation*

$$y^{(4)} - y''' - y'' + y' = t^2 + 4 + t\sin t.$$

We need to determine the solutions to the homogeneous equation so that our guess does not include a multiple of one of these solutions. The characteristic equation for

$$y^{(4)} - y''' - y'' + y' = 0$$

is

$$r^4 - r^3 - r^2 + r = r(r^3 - r^2 - r + 1) = 0.$$

By Newton's divisibility criterion, the possible rational roots of above cubic are ± 1, and, in fact, $r = \pm 1$ are both roots. Therefore, by long division characteristic equation is

$$r(r-1)(r+1) \cdot x = r^4 - r^3 - r^2 + r = 0.$$

By long division of polynomials, we see that

$$x = r - 1.$$

Therefore, the characteristic equation is

$$r(r-1)^2(r+1) = 0.$$

So the solutions to the homogeneous equation are

$$1, \qquad e^t, \qquad te^t, \qquad e^{-t}.$$

We can write the particular solution to

$$y^{(4)} - y''' - y'' + y' = t^2 + 4 + t\sin t$$

as the sum of solutions to the differential equations

$$y^{(4)} - y''' - y'' + y' = t^2 + 4$$

and

$$y^{(4)} - y''' - y'' + y' = t\sin t.$$

Our first guess for the particular solution to the nonhomogeneous equation

$$y^{(4)} - y''' - y'' + y' = t^2 + 4$$

is

$$Y(t) = A_1 t^2 + A_2 t + A_3.$$

We must ask ourselves, are any of the terms above multiples of the solutions

$$1, \qquad e^t, \qquad te^t, \qquad e^{-t},$$

to the homogeneous equation?

Yes, the constant term, involving only A_3, is a multiple of the solution

$$y_1(t) = 1$$

to the homogeneous equation.

Therefore, we need to multiply our guess by t before we start taking derivatives:

$$Y(t) = A_1 t^3 + A_2 t^2 + A_3 t$$

Our guess for the particular solution to

$$y^{(4)} - y''' - y'' + y' = t \sin t$$

is

$$Y(t) = (B_1 t + B_2) \cos t + (C_1 t + C_2) \sin t.$$

None of the terms in our guess is a multiple of any of the solutions

$$1, \qquad e^t, \qquad te^t, \qquad e^{-t}$$

to the homogeneous equation.

Therefore, our guess for the particular solution to

$$y^{(4)} - y''' - y'' + y' = t^2 + 4 + t \sin t$$

is

$$Y(t) = A_1 t^3 + A_2 t^2 + A_3 t + (B_1 t + B_2) \cos t + (C_1 t + C_2) \sin t.$$

For a complicated function of the above form, involving many unknown parameters, a computer algebra system can be extremely helpful in executing the tedious algebraic calculations needed to apply the method of undetermined coefficients.