Higher Order Homogeneous Differential Equations with Constant Coefficients

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We know how to solve equations of the form

$$ay'' + by' + cy = 0, \qquad a, b, c, \in \mathbb{R}.$$

How do we solve similar equations with higher order derivatives?

Recall that in the second order case, we assumed that $y(t) = e^{rt}$ was a solution for some suitable value of r.

We showed that, in order for $y(t) = e^{rt}$ to be a solution to

$$ay'' + by' + cy = 0,$$

we needed to have

$$ar^2 + br + c = 0.$$

So, what does r have to satisfy in order for $y(t) = e^{rt}$ to be a solution to

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0 \qquad ?$$

By plugging $y(t) = e^{rt}$ into this differential equation, and then factoring out an e^{rt}, we find that

$$e^{rt}(a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n) = 0,$$

so that

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0.$$

The equation

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$$

is called the **characteristic equation** for the differential equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0.$$

Theorem. *If we are permitted to use complex numbers, then a polynomial of degree* n*, say*

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n$$

always has n *zeros, some of which may be repeated.*

Therefore, the characteristic polynomial can be written

$$a_0 (r - r_1)(r - r_2) \cdots (r - r_n),$$

where

$$r_1, r_2, \ldots, r_n$$

are the roots of the characteristic polynomial.

As in the order 2 case, we need to know how to handle the cases where:

1. The roots are all distinct and real,
2. Roots are repeated,
3. Some of the roots are complex.

Theorem. *If the roots* $r_1, r_2, \ldots, r_n$ *of the characteristic equation*

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$$

are all real, and no two are equal, then the functions

$$e^{r_1 t}, e^{r_2 t}, \cdots, e^{r_n t}$$

form a fundamental set of solutions to

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0.$$

Although we can always factor a polynomial using complex numbers, there is no "nice" formula, like the quadratic formula, for polynomials of degree 5 and greater.

We can try to factor particular polynomials (of any degree) using techniques that work in special cases.

Newton's divisibility Criterion. Suppose the polynomial

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$$

has integer coefficients. If $r = p/q$ is a solution, then p must be a factor of a_n, and q must be a factor of a_0.

Example. *What are the possible rational roots of*

$$r^4 + r^3 - 7r^2 - r + 6 \qquad ?$$

The factors of the *constant* coefficient, 6, are

$$\pm 1, \pm 2, \pm 3, \pm 6.$$

The factors of the *leading* coefficient, 1 are

$$\pm 1.$$

Newton's criterion tells us that the only possible rational roots are

$$\frac{\pm 1}{\pm 1}, \quad \frac{\pm 2}{\pm 1}, \quad \frac{\pm 3}{\pm 1}, \quad \frac{\pm 6}{\pm 1} \quad = \quad \pm 1, \quad \pm 2, \quad \pm 3, \quad \pm 6.$$

By plugging in each of these eight numbers, we find that

$$r = 1, -1, 2, -3$$

satisfy

$$r^4 + r^3 - 7r^2 - r + 6 = 0.$$

Therefore, we can write

$$r^4 + r^3 - 7r^2 - r + 6 = (r-1)(r+1)(r-2)(r+3).$$

It follows (because the roots are real and distinct) that

$$e^t, \qquad e^{-t}, \qquad e^{2t}, \qquad e^{-3t}$$

form a fundamental set of solutions to

$$y^{(4)} + y^{(3)} - 7y'' - y' + 6y = 0.$$

Example. *Solve the initial value problem*

$$y^{(4)} + y^{(3)} - 7y'' - y' + 6y = 0,$$

$$y(0) = 1, \qquad y'(0) = 0, \qquad y''(0) = -2, \qquad y^{(3)}(0) = -1.$$

From the last example, we know that that the general solution is

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-3t}.$$

Differentiating, we find that

$$\begin{aligned} y'(t) &= c_1 e^t - c_2 e^{-t} + 2c_3 e^{2t} - 3c_4 e^{-3t}, \\ y''(t) &= c_1 e^t + c_2 e^{-t} + 4c_3 e^{2t} + 9c_4 e^{-3t}, \\ y^{(3)}(t) &= c_1 e^t - c_2 e^{-t} + 8c_3 e^{2t} - 27c_4 e^{-3t}. \end{aligned}$$

Therefore, the initial conditions imply that

$$\begin{aligned} 1 &= y(0) = c_1 + c_2 + c_3 + c_4, \\ 0 &= y'(0) = c_1 - c_2 + 2c_3 - 3c_4, \\ -2 &= y''(0) = c_1 + c_2 + 4c_3 + 9c_4, \\ -1 &= y^{(3)}(0) = c_1 - c_2 + 8c_3 - 27c_4. \end{aligned}$$

We can solve this system (for example, by Cramer's rule – or on a computer) to find that

$$c_1 = \frac{11}{8}, \qquad c_2 = \frac{5}{12}, \qquad c_3 = -\frac{2}{3}, \qquad c_4 = -\frac{1}{8}.$$

What if we have a polynomial in which some of the roots are complex?

Theorem. *If the characteristic equation involving a polynomial with real coefficients has complex roots, then they occur in conjugate pairs* $\lambda \pm \mu i$. *In this case, we can replace the complex valued solutions*

$$e^{(\lambda+\mu i)t}, \qquad e^{(\lambda-\mu i)t}$$

with the real-valued functions

$$e^{\lambda t}\cos\mu t, \qquad e^{\lambda t}\sin\mu t.$$

How about the case where roots are repeated?

Theorem. *If the number* r_1 *is repeated* k *times as a root of the characteristic equation, then*

$$e^{r_1 t}, \qquad te^{r_1 t}, \qquad t^2 e^{r_1 t}, \quad \cdots \quad , \qquad t^k e^{r_1 t}$$

are corresponding solutions.

Sometimes *complex* roots are repeated, so we have to combine the above two theorems.

Example. *Find the general solution to*

$$y^{(4)} - y = 0.$$

The characteristic equation is

$$r^4 - 1 = 0.$$

Using the formula for a difference of squares (twice!), we find that

$$0 = r^4 - 1 = (r^2 - 1)(r^2 + 1) = (r - 1)(r + 1)(r + i)(r - i).$$

Therefore, the general solution is

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t.$$

- We get the first term from the fact that 1 is a root of the characteristic equation.
- We get the second term from the fact that -1 is a root,
- We get the third term from the fact that i is a root.
- We get the fourth term from the fact that $-i$ is a root.

Example. *Find the general solution to*

$$y^{(4)} + 2y'' + y = 0.$$

The characteristic equation is

$$r^4 + 2r^2 + 1 = (r^2 + 1)^2 = [(r + i)(r - i)]^2 = 0.$$

Thus, the roots are complex and repeated, namely,

$$r = i,\ i,\ -i,\ -i.$$

The general solution is therefore

$$y(t) = c_1 \cos t + c_2 t \cos t + c_3 \sin t + c_4 t \sin t.$$

- We get the first term from the fact that $i = 0 + i = \lambda + \mu i$ is a root.
- We get the second term from the fact that i appears twice as a root.
- We get the third term from the fact that $-i$ is a root.
- We get the fourth term from the fact that $-i$ appears twice as a root.

Example. *Find the general solution to*

$$y^{(4)} - 8y' = 0.$$

The characteristic equation is

$$r^4 - 8r = 0.$$

We can factor to get

$$r(r^3 - 8) = 0.$$

Using the formula for a difference of cubes from algebra, we can write the characteristic equation as

$$r(r-2)(r^2 + 2r + 4) = 0.$$

The roots of

$$r^2 + 2r + 4 = 0$$

are

$$r = \frac{-2 \pm \sqrt{4-16}}{2} = -1 \pm \frac{\sqrt{-12}}{2} = -1 \pm \sqrt{3}i.$$

Therefore, the roots of the characteristic equation are

$$r = 0,\ 2,\ -1 + \sqrt{3}i,\ -1 - \sqrt{3}i.$$

The general solution to the differential equation is therefore,

$$y(t) = c_1 + c_2 e^{2t} + c_3 e^{-t} \cos(\sqrt{3}t) + c_4 e^{-t} \sin(\sqrt{3}t).$$