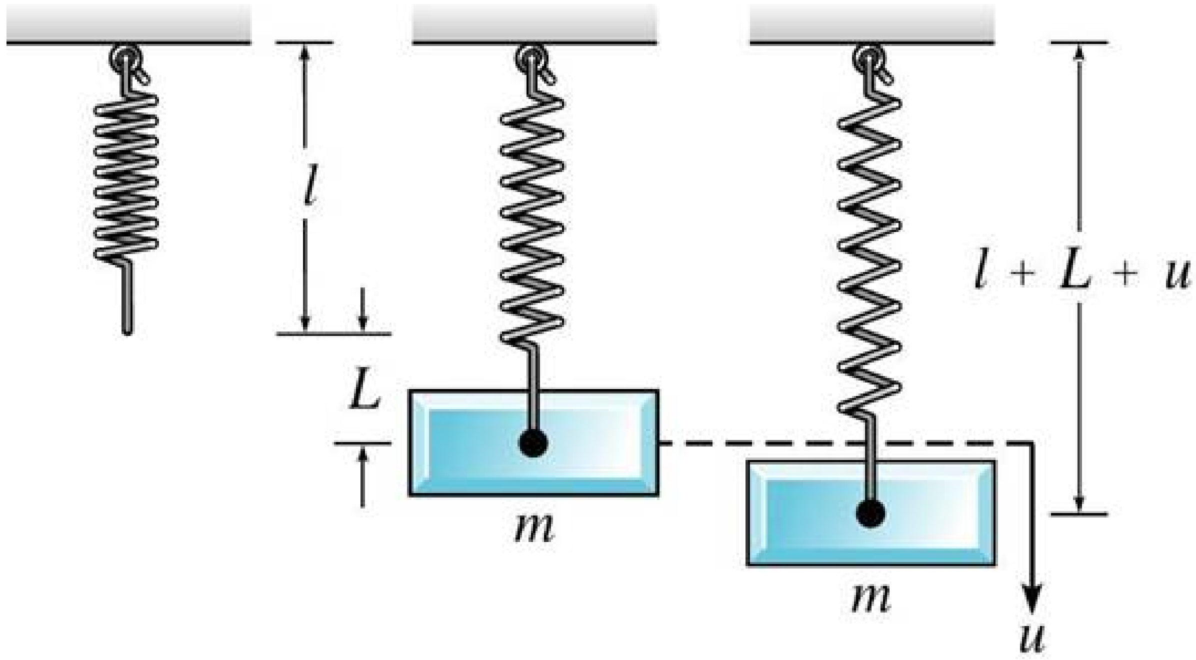
Mechanical Vibrations

Linear second-order differential equations model the oscillatory motion of a spring.

The same principles we apply here can be used to model other systems involving vibrations or periodic oscillations

Examples:

- Acoustics
- The vibration of a bridge under external forces such as traffic and wind
- Electrical current



Definitions:

- l is the length of the spring.
- L is the additional elongation caused by the mass.
- $u(t)$ is the displacement from the original position at time t.

Key facts from physics:

Newton's Second Law of Motion:

$$F = ma$$

Force = Mass × Acceleration.

Hooke's Law:
The force exerted by the spring, F_s on the mass (directed upward) is proportional to L, the elongation due to the weight of the mass:

$$F_s = -kL,$$

The spring force acts **upwards**, in the **negative** direction.

If $u(t)$ represents a **position function** at time t, then

$u'(t)$ represents the corresponding **velocity function**,

and $u''(t)$ is the corresponding **acceleration function**.

If we denote the magnitude of external forces in the spring system by $f(t)$, then Newton's Second Law of motion implies that a model for the system is

$$\underbrace{f(t)}_{\text{force}} \qquad = \qquad \underbrace{m \quad \times \quad u''(t)}_{\text{mass} \times \text{acceleration}}$$

To make the model precise, we need to account for each force acting on the system.

And the forces are:

- If m is the **mass** of the weight, then, by Newton's second law, the weight exerts a downward (positive magnitude) force of

  $$w = mg,$$

  where g is the acceleration due to gravity.

- Using Hooke's law, the **force exerted by the spring** at times t is

  $$F_s(t) = -k[L + u(t)],$$

  where k is some positive constant of proportionality.

  Note that when $L + u(t) < 0$ (the spring is compressed), the resulting force $F_s(t)$ is positive and so exerted $\downarrow$.

- The **damping** or resistive force $F_d(t)$, caused, say, by air resistance or friction is assumed to be proportional to the velocity of the mass at time t:

  $$F_d(t) = -\gamma u'(t),$$

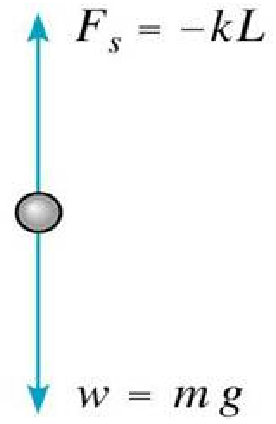
  where γ is some positive constant of proportionality. Note that if the velocity is in the negative $\uparrow$ direction, then the resistive force is positive, so directed $\downarrow$.

- We also account for the possibility of other variable **external forces** by $F(t)$.

So our model is

$$\begin{aligned} mu''(t) &= mg + F_s(t) + F_d(t) + F(t) \\ &= mg - k[L + u(t)] - \gamma u'(t) + F(t). \end{aligned}$$

Since the force of the spring with the weight attached (at rest) is equal and opposite to the force exerted by the weight,



we have

$$mg - kL = 0,$$

so the model simplifies to

$$mu''(t) + \gamma u'(t) + ku(t) = F(t),$$

where

- m is the mass of the suspended object,
- γ is the damping constant,
- k is the spring constant,
- $F(t)$ represents all other external forces.

Let us first consider a simple case where there are no other external forces and no damping, so

$$F(t) = 0 \qquad \text{and} \qquad \gamma = 0.$$

The model above simplifies to

$$mu''(t) + ku(t) = 0.$$

The corresponding characteristic equation is

$$mr^2 + k = 0.$$

The roots of the characteristic equation are

$$r = \pm\sqrt{\frac{k}{m}}i.$$

Therefore, the general solution to the differential equation is

$$\begin{aligned} u(t) &= A\cos\left(\sqrt{\frac{k}{m}} \cdot t\right) + B\sin\left(\sqrt{\frac{k}{m}} \cdot t\right) \\ &= A\cos\omega_0 t + B\sin\omega_0 t. \end{aligned}$$

where we denote the constant $\sqrt{\frac{k}{m}}$ by ω_0, and call it the **circular frequency**.

We will now rewrite the solution $u(t)$ using new parameters R and δ, defined in terms of the numbers above by

$$A = R\cos\delta \qquad \text{and} \qquad B = R\sin\delta.$$

In other words, we find δ and R from A and B via the formulas

$$\frac{B}{A} = \frac{\sin\delta}{\cos\delta} = \tan\delta \qquad \text{and} \qquad R = \sqrt{A^2 + B^2}.$$

We can then re-write $u(t)$ as

$$\begin{aligned} u(t) &= R\cos\delta\cos\omega_0 t + R\sin\delta\sin\omega_0 t \\ &= R\Big(\cos\delta\cos\omega_0 t + \sin\delta\sin\omega_0 t\Big) \\ &= R\cos(\omega_0 t - \delta), \end{aligned} \tag{1}$$

where, to get the last equality, we applied the trigonometric identity

$$\cos(x-y) = \cos x\cos y + \sin x\sin y.$$

In the solution

$$u(t) = R\cos(\omega_0 t - \delta), \tag{2}$$

the parameter R is called the **amplitude** of the motion, and measures the magnitude of oscillation.

The parameter δ is called the **phase angle**. This parameter measures the displacement of the wave from its normal position corresponding to $\delta = 0$.

Notice that

$$u(t) = R\cos(\omega_0 t - \delta)$$

is just a shifted cosine curve with a **period**[a], T, given by

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi\left(\frac{m}{k}\right)^{1/2}.$$

Notice that if the mass is large, then T is large, so that the spring will vibrate more slowly. If k is large, then T will decrease, so a more rigid spring will cause the system to vibrate faster.

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[a] T is the amount of time it takes for the system to undergo a complete cycle.

Example. *A mass weighing* 5 *pounds stretches a spring* 3 *inches. Suppose that the mass is forced an additional* 6 *inches down and then released with an downward velocity of* $1ft/sec$*. Determine the position of the mass at all later times. Also determine the period,* T*, amplitude,* R*, and phase,* δ*, of the motion.*

Since the mass stretches the spring 3 inches $= \frac{1}{4}$ foot, and the force exerted by the mass is 5 pounds, we have

$$F = kL \qquad \Longrightarrow \qquad 5 = \frac{1}{4}k \qquad \Longrightarrow k = 20 \text{ lb/ft}.$$

Therefore, the model for this system is (with $g = 32$ ft/sec $\implies m = \frac{w}{g}$)

$$\frac{5}{32}u'' + 20u = 0.$$

The roots of the characteristic equation are

$$r = \pm\sqrt{128} \cdot i.$$

The general solution to the differential equation is

$$u(t) = A\cos\left(\sqrt{128} \cdot t\right) + B\sin\left(\sqrt{128} \cdot t\right).$$

The initial conditions (in feet) are $u(0) = \frac{1}{2},\ u'(0) = 1$

$$\Longrightarrow \qquad A = \frac{1}{2} \qquad \text{and} \qquad B = \frac{\sqrt{2}}{16}.$$

This means

$$\delta = \arctan\frac{B}{A} \approx 0.174969, \qquad R = \sqrt{A^2 + B^2} \approx 0.507752, \qquad T = \frac{2\pi}{\sqrt{128}}.$$

Using these approximations for δ and R, we have

$$u(t) \approx 0.507752\cos\left(\sqrt{128} \cdot t - 0.174969\right).$$

With a nonzero damping constant, the model for the system becomes

$$mu''(t) + \gamma u'(t) + ku(t) = 0.$$

The corresponding characteristic equation is

$$mr^2 + \gamma r + k = 0,$$

and the roots are

$$\begin{aligned} r_1, r_2 &= \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} \\ &= \frac{-\gamma \pm \sqrt{\gamma^2 \left(1 - \frac{4km}{\gamma^2}\right)}}{2m} \\ &= -\frac{\gamma \pm \gamma\sqrt{1 - \frac{4km}{\gamma^2}}}{2m} \\ &= \frac{-\gamma}{2m}\left(1 \pm \sqrt{1 - \frac{4km}{\gamma^2}}\right). \end{aligned}$$

Notice that the *kind* of solution we get depends on the sign of $\gamma^2 - 4km$.

For nonnegative values of $\gamma^2 - 4km$, we have

$$\begin{aligned} u(t) &= Ae^{r_1 t} + Be^{r_2 t}, \qquad &&\text{if } \gamma^2 - 4km > 0, \\ u(t) &= (A + Bt)e^{-\gamma t/(2m)}, \qquad &&\text{if } \gamma^2 - 4km = 0, \end{aligned}$$

and, if $\gamma^2 - 4km < 0$,

$$u(t) = e^{-\gamma t/(2m)}(A \cos \mu t + B \sin \mu t), \qquad \mu = \frac{(4km - \gamma^2)^{1/2}}{2m}.$$

Notice how the nature of the solution changes as $\gamma^2 - 4km$ changes sign. The point where this happens, $\gamma = 2\sqrt{km}$ is called the **critical damping**.

Note that

$$4km > 0 \quad \Longrightarrow \quad -4km < 0 \quad \Longrightarrow \quad \gamma^2 - 4km < \gamma^2.$$

So,

- If r_1, r_2 are **real**, then the numerators of the values
$$\frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$$
are negative, so that $r_1, r_2 < 0$.
- If r_1, r_2 are **complex** $= \lambda \pm i\mu$, then $\lambda = -\frac{\gamma}{2m} < 0$.

Therefore, in all cases, we have exponential decay dominating each solution. This means that the damping gradually dissipates the energy in the system.

For example, if r_1 and r_2 are complex, then the solution is

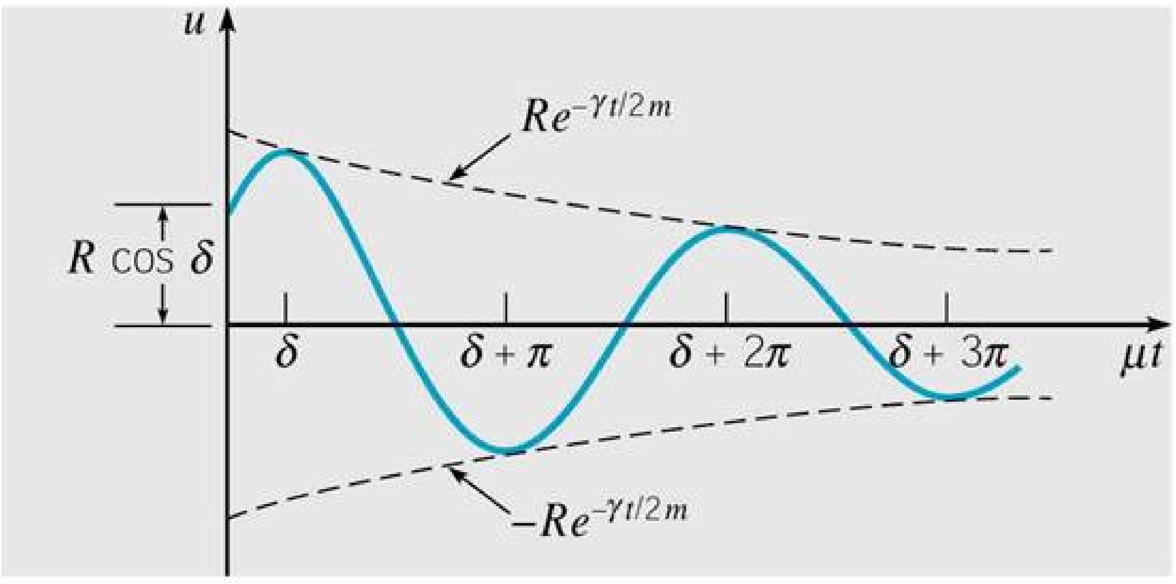
$$u(t) = e^{-\gamma t/(2m)}(A\cos\mu t + B\sin\mu t),$$

which can be written, via the relations

$$A = R\cos\delta \qquad \text{and} \qquad B = R\sin\delta$$

as

$$u(t) = Re^{-\gamma t/(2m)}\cos(\mu t - \delta).$$



In this case, where

$$u(t) = Re^{-\gamma t/(2m)} \cos(\mu t - \delta), \qquad \mu = \frac{(4km - \gamma^2)^{1/2}}{2m},$$

we call the constant μ the **quasi frequency** and the corresponding value

$$T_d = \frac{2\pi}{\mu}$$

the **quasi period**.