Variation of Parameters

Goal: Find a formula for a solution to a general nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

in terms of the functions $p(t)$, $q(t)$, and $g(t)$ and the functions y_1 and y_2, the linearly independent solutions to

$$y'' + p(t)y' + q(t)y = 0.$$

Idea: Start with the general solution

$$c_1 y_1(t) + c_2 y_2(t), \qquad c_1, c_2 \in \mathbb{C}$$

to the corresponding homogeneous equation

$$y'' + p(t)y' + q(t)y = 0.$$

Replace the constants c_1 and c_2 by functions of t and determine if the requirement that the resulting function satisfies the nonhomogeneous differential equation

$$y'' + p(t)y' + q(t)y = g(t)$$

provides information about the functions.

> We replace the constants with functions that change with t, and so "vary the parameters".

We will determine conditions on $u_1(t)$ and $u_2(t)$ so that

$$Y(t) = \underbrace{u_1(t)}_{\text{"New"}c_1}\, y_1(t) + \underbrace{u_2(t)}_{\text{"New"}c_2}\, y_2(t)$$

is a solution to $y'' + p(t)y' + q(t)y = g(t)$.

If we differentiate

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t),$$

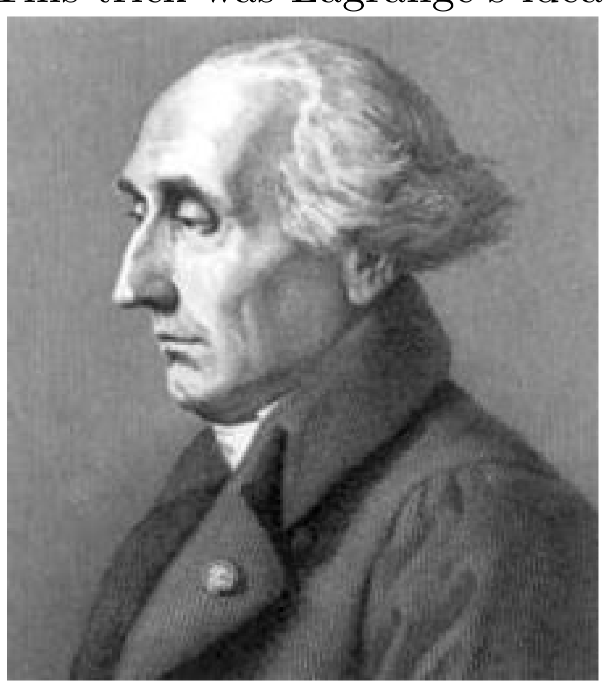
we get

$$Y'(t) = u_1'(t)y_1(t) + u_1(t)y_1'(t) + u_2'(t)y_2(t) + u_2(t)y_2'(t).$$

To simplify things, we will impose the requirement that

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0.$$

This trick was Lagrange's idea.



Joseph Louis Lagrange (1736 - 1813)

Therefore, we have

$$Y'(t) = u_1(t)y_1'(t) + u_2(t)y_2'(t).$$

Taking the derivative of $Y'(t)$ gives

$$Y''(t) = u_1'(t)y_1'(t) + u_1(t)y_1''(t) + u_2'(t)y_2'(t) + u_2(t)y_2''(t).$$

We now plug the expressions $Y(t), Y'(t)$, and $Y''(t)$ into the differential equation we would like $Y(t)$ to satisfy

$$y'' + p(t)y' + q(t)y = g(t).$$

We get

$$\begin{aligned}u_1'(t)y_1'(t) + u_1(t)y_1''(t) + u_2'(t)y_2'(t) + u_2(t)y_2''(t)\\ + p(t)\Big(u_1(t)y_1'(t) + u_2(t)y_2'(t)\Big)\\ + q(t)\Big(u_1(t)y_1(t) + u_2(t)y_2(t)\Big) = g(t).\end{aligned}$$

Let's now group all of the terms involving y_1 together, and do the same for y_2:

$$\begin{aligned}u_1(t)\Big(y_1''(t) + p(t)y_1'(t) + q(t)y_1(t)\Big)\\ + u_2(t)\Big(y_2''(t) + p(t)y_2'(t) + q(t)y_2(t)\Big)\\ + u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t).\end{aligned}$$

Since y_1 and y_2 are solution to the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0,$$

we can reduce the expression above to

$$\begin{aligned}u_1(t)\cdot 0\\ + u_2(t)\cdot 0\\ + u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t).\end{aligned}$$

Therefore, we need to find $u_1(t)$ and $u_2(t)$ so that

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t).$$

If we put this last equation together with Lagrange's requirement, we need $u_1(t)$ and $u_2(t)$ to satisfy

$$\begin{cases} u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0,\\ u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t)\end{cases}$$

We have two equations, two unknowns ($u_1'(t)$ and $u_2'(t)$), so we can solve this system to find

$$u_1'(t) = \frac{-y_2(t)g(t)}{y_1(t)y_2'(t) - y_2(t)y_1'(t)},$$
$$u_2'(t) = \frac{y_1(t)g(t)}{y_1(t)y_2'(t) - y_2(t)y_1'(t)}.$$

Since the denominator in each expression is equal to

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t),$$

we see that

$$u_1'(t) = \frac{-y_2(t)g(t)}{W(y_1, y_2)(t)},$$
$$u_2'(t) = \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}.$$

To finally find $u_1(t)$ and $u_2(t)$ explicitly, we integrate both sides of these equations:

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)} dt + C_1,$$
$$u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} dt + C_2.$$

If we can find these integrals, then our solution to

$$y'' + p(t)y' + q(t)y = g(t).$$

is

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t).$$

Theorem. *If $p(t)$ and $q(t)$ are continuous on the open interval I, and if y_1 and y_2 are two linearly independent solutions to the homogeneous equation*

$$y'' + p(t)y' + q(t)y = 0,$$

then a solution to the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

is given by

$$Y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2)(s)}ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2)(s)}ds,$$

where t_0 is any point in I.

The general solution to the nonhomogeneous equation is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t).$$

Example: Find the general solution to

$$y'' + y = \tan t, \qquad 0 < t < \pi/2.$$

We first solve the corresponding homogeneous equation

$$y'' + y = 0.$$

The characteristic equation is $r^2 + r = 0$, so that

$$r = \frac{-0 \pm \sqrt{-4}}{2} = 0 \pm \frac{\sqrt{4}}{2}i = 0 \pm i = \pm i.$$

Therefore, $y_1(t) = \cos t$ and $y_2(t) = \sin t$ are the linearly independent solutions to the *homogeneous* equation.

The Wronskian of these two solutions is

$$W(y_1, y_2)(t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1.$$

From the method of variation of parameters, a particular solution to the nonhomogeneous equation is

$$Y(t) = -\cos t \int_{\pi/4}^{t} \frac{(\sin s)(\tan s)}{\underbrace{1}_{W(y_1,y_1)(t)}} ds + \sin t \int_{\pi/4}^{t} \frac{(\cos s)(\tan s)}{\underbrace{1}_{W(y_1,y_1)(t)}} ds.$$

(Why did we use $\pi/4$ as the lower limit of integration? Because the number $\pi/4$ lies in the interval $0 < t < \pi/2$, where we want to solve the equation. We could have chosen any other point in this interval as the lower limit of integration.)

How are we going to find the integrals appearing in our expression for $Y(t)$?

We simplify the trigonometric quotient appearing in the integrand.

$$\begin{aligned} \int_{\pi/4}^{t} (\sin s)(\tan s)\ ds &= \int_{\pi/4}^{t} \frac{sin^2 s}{\cos s} ds \\ &= \int_{\pi/4}^{t} \frac{1 - \cos^2 s}{\cos s} ds \\ &= \int_{\pi/4}^{t} \sec s\ ds - \int_{\pi/4}^{t} \cos s\ ds \\ &= \ln(\sec s + \tan s)\Big|_{s=\pi/4}^{s=t} - \sin s\Big|_{s=\pi/4}^{s=t} \\ &= \ln(\sec t + \tan t) - \ln(\sqrt{2} + 1) - \sin t + \frac{\sqrt{2}}{2}. \end{aligned}$$

The second integral in the expression for $Y(t)$ is much easier to compute:

$$\int_{\pi/4}^{t} (\cos s)(\tan s)\ ds = \int_{\pi/4}^{t} \sin s\ ds = -\cos s\Big|_{s=\pi/4}^{s=t} = -\cos t + \frac{\sqrt{2}}{2}.$$

Therefore,

$$\begin{aligned} Y(t) &= (-\cos t)\Big(\ln(\sec t + \tan t) - \ln(\sqrt{2}-1) - \sin t + \frac{\sqrt{2}}{2}\Big) \\ &\quad + (\sin t)\Big(\frac{\sqrt{2}}{2} - \cos t\Big) \leftarrow \boxed{\text{The terms in red cancel after expanding each expression}}. \\ &= (-\cos t)\ln(\sec t + \tan t) + \underbrace{\left(\frac{\sqrt{2}}{2} - \ln(\sqrt{2}+1)\right)\cos t + \frac{\sqrt{2}}{2}\sin t}_{\text{This is a solution to the homogeneous equation. When we plug it into the differential equation, we get 0.}} \end{aligned}$$

So it suffices to take

$$Y(t) = -(\cos t)\ln(\sec t + \tan t)$$

as the particular solution to $y'' + y = \tan t$.

The general solution to $y'' + y = \tan t$ is

$$y(t) = c_1 \cos t + c_2 \sin t - (\cos t)\Big(\ln(\sec t + \tan t)\Big).$$