Nonhomogeneous Equations and the Method of Undetermined Coefficients

We now connect our study of homogeneous equations

$$y'' + p(t)y' + q(t)y = 0$$

to the nonhomogeneous case.

What do we do with more general differential equations, like

$$y'' + p(t)y' + q(t)y = g(t),$$

where $g(t)$ is a function not identically equal to zero?

It turns out that solutions to the nonhomogeneous equation are connected to solutions of the homogeneous equation.

Theorem. *If Y_1 and Y_2 are two solutions to the nonhomogeneous equation*

$$y'' + p(t)y' + q(t)y = g(t),$$

then their difference $Y_1 - Y_2$ is a solution to the corresponding homogeneous equation

$$y'' + p(t)y' + q(t)y = 0. \tag{1}$$

If, in addition, we have two linearly independent solutions y_1 and y_2 to equation (1), then

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t)$$

for some appropriate choice of constants c_1 and c_2.

Proof. Suppose that Y_1 and Y_2 are two solutions to the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t).$$

Then since

$$Y_1'' + p(t)Y_1' + q(t)Y_1 = g(t) \qquad \text{and} \qquad Y_2'' + p(t)Y_2' + q(t)Y_2 = g(t),$$

We see that

$$\begin{aligned}
&(Y_1 - Y_2)'' + p(t)(Y_1 - Y_2)' + q(t)(Y_1 - Y_2) \\
&\qquad = Y_1'' - Y_2'' + p(t)Y_1' - p(t)Y_2' + q(t)Y_1 = q(t)Y_2 \\
&\qquad = \Big[Y_1'' + p(t)Y_1' + q(t)Y_1\Big] - \Big[Y_2'' + p(t)Y_2' + q(t)Y_2\Big] \\
&\qquad = g(t) - g(t) = 0.
\end{aligned}$$

This proves the first part of the theorem.

Since $Y_1(t) - Y_2(t)$ is a solution to the *homogeneous* equation, we may express it as a linear combination of two linearly independent solutions $y_1(t)$ and $y_2(t)$ to the corresponding *homogeneous* equation, for some appropriate choice of constants.

In other words,

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t)$$

for some constants c_1 and c_2. $\square$

This brings us to the following important theorem.

Theorem. *The general solution to the nonhomogeneous equation*

$$y'' + p(t)y' + q(t)y = g(t) \tag{2}$$

can be written in the form

$$y(t) = c_1y_1(t) + c_2y_2(t) + Y(t),$$

where y_1 and y_2 are a fundamental set of solutions to the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0, \tag{3}$$

c_1 and c_2 are arbitrary constants, and $Y(t)$ is any solution to the nonhomogeneous equation (2).

Proof. If y_1 and y_2 form a fundamental set of solutions for the homogeneous equation (3) and $Y(t)$ is a solution to the nonhomogeneous equation (2), then

$$y(t) = c_1y_1(t) + c_2y_2(t) + Y(t)$$

is a solution of (2) for any choice of constants since

$$\begin{aligned}
y'' + p(t)y' + q(t)y &= (c_1y_1(t) + c_2y_2(t) + Y(t))'' \\
&\qquad + p(t)(c_1y_1(t) + c_2y_2(t) + Y(t))' \\
&\qquad\qquad + q(t)(c_1y_1(t) + c_2y_2(t) + Y(t)) \\
&= c_1(y_1'' + p(t)y_1' + q(t)y_1) \qquad \longleftarrow 0 \\
&\qquad + c_2(y_2'' + p(t)y_2' + q(t)y_2) \qquad \longleftarrow 0 \\
&\qquad\qquad + (Y'' + p(t)Y' + q(t)Y) \qquad \longleftarrow g(t) \\
&= 0 + 0 + g(t) = g(t).
\end{aligned}$$

We now know that

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

is a solution.

Why is it the general solution?

Let $\phi(t)$ be some arbitrary solution to the *nonhomogeneous* equation with y_1 and y_2 a fundamental set of solutions for the *homogeneous* equation. We need to show that $\phi(t)$ has a representation of the form

$$c_1 y_1(t) + c_2 y_2(t) + Y(t).$$

Since $Y(t)$ and $\phi(t)$ are both solutions of the *nonhomogeneous* equation, their difference is a solution to the *homogeneous* equation. The theorem on page one implies that, for some appropriate choice of c_1 and c_2,

$$\phi(t) - Y(t) = c_1 y_1(t) + c_2 y_2(t).$$

Therefore, ϕ is representable as a linear combination of y_1, y_2 and $Y(t)$, so we are justified in calling the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

the general solution to the *nonhomogeneous* equation

$$y'' + p(t)y' + q(t)y = g(t).$$

□

How do we apply this theorem to solve a nonhomogeneous differential equation

$$y'' + p(t)y' + q(t)y = g(t) \quad ?$$

We need to do three things:

1. Find the general solution
$$c_1y_1(t) + c_2y_2(t)$$
to the corresponding homogeneous equation
$$y'' + p(t)y' + q(t)y = 0.$$
2. Find some particular solution $Y(t)$ to the nonhomogeneous equation
$$y'' + p(t)y' + q(t)y = g(t).$$
3. Add together the functions from steps 1 and 2.

The difficulty lies in step 2.

How do we find some particular solution to

$$y'' + p(t)y' + q(t)y = g(t) \quad ?$$

One technique we can use is called the **method of undetermined coefficients.**

The Method of Undetermined Coefficients

Idea: We guess at the form of the particular solution $Y(t)$ to the nonhomogeneous differential equation

$$y'' + p(t)y' + q(t)y = g(t),$$

but without specifying the coefficients.

Example. *Find a particular solution to the differential equation*

$$y'' - 3y' - 4y = 3e^{2t} \tag{4}$$

and use this to find the general solution.

We know that the derivative of an exponential function is some number times the original function.

It would be reasonable to assume that we could find a solution of the form

$$Y(t) = Ae^{2t}.$$

In this case,

$$Y'(t) = 2Ae^{2t}, \qquad Y''(t) = 4Ae^{2t}.$$

Plugging these calulations into (4), we get

$$(\underbrace{4Ae^{2t}}_{Y''(t)}) - 3(\underbrace{2Ae^{2t}}_{Y'(t)}) - 4\underbrace{Ae^{2t}}_{Y(t)} = 3e^{2t}.$$

In other words,

$$4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} = 3e^{2t},$$

or equivalently

$$-6Ae^{2t} = 3e^{2t} \qquad \Longrightarrow \qquad A = -\frac{1}{2}.$$

Therefore,

$$Y(t) = -\frac{1}{2}e^{2t}$$

is a solution to the nonhomogeneous differential equation

$$y'' - 3y' - 4y = 3e^{2t}.$$

Note that the general solution to the corresponding homogeneous equation

$$y'' - 3y' - 4y = 0.$$

is

$$c_1e^{4t} + c_2e^{-t}.$$

Therefore, by the theorem above, the general solution to the nonhomogeneous equation

$$y'' - 3y' - 4y = 3e^{2t}$$

is given by

$$y(t) = c_1e^{4t} + c_2e^{-t} - \frac{1}{2}e^{2t}.$$

Example. *Find a particular solution to*

$$y'' - 3y' - 4y = 2e^{-t}. \tag{5}$$

If we assume that $Y(t) = Ae^{-t}$ is a solution and proceed as in the last example, we obtain

$$Ae^{-t} + 3Ae^{-t} - 4Ae^{-t} = 2e^{-t}.$$

But here the left side reduces to zero. That is, we have

$$0 = 2e^{-t}.$$

This happened because (as we saw on the last page)

$$y(t) = e^{-t}$$

is a solution to the homogeneous differential equation.

In other words, it makes the left side of (5) zero; that is

$$y'' - 3y' - 4y = 0.$$

We therefore need to be more industrious in finding the form of our particular solution $Y(t)$ to the nonhomogeneous equation.

Since e^{-t} is a solution to the corresponding homogeneous equation, let us suppose that $Y(t) = v(t)e^{-t}$ and use the method of **reduction of order** to find a function $v(t)$ that works.

From the assumption

$$Y(t) = v(t)e^{-t}, \tag{6}$$

we use the product rule to find $Y'(t)$, $Y''(t)$, and plugging these values back into the original differential equation,

$$y'' - 3y' - 4y = 2e^{-t}, \tag{7}$$

We find (omitting a lot of work) that

$$v'' - 5v' = 2.$$

Leting $w(t) = v'(t)$, this equation reduces to

$$w' - 5w = 2. \tag{8}$$

Using the method of integration factors, we find that the general solution to (8) is

$$w(t) = -\frac{2}{5} + ce^{5t}, \qquad c \in \mathbb{R}.$$

In other words,

$$v'(t) = -\frac{2}{5} + ce^{5t}.$$

Integrating, we find that

$$v(t) = -\frac{2}{5}t + \frac{c}{5}e^{5t} + k.$$

Finally, using assumption (6) at the top of the page, we see that

$$Y(t) = e^{-t}v(t) = \underbrace{-\frac{2}{5}te^{-t}}_{\text{keep}} + \underbrace{\frac{c}{5}e^{4t} + ke^{-t}}_{\text{throw out}}.$$

We throw out the last two terms in $Y(t)$ since we know that they are solutions to the homogeneous equation.

Therefore,

$$Y(t) = -\frac{2}{5}te^{-t}$$

is a particular solution to the nonomogeneous differential equation.

Notice that the solution is a function of the form Ate^{-t}, which is t times what our initial guess was.

A Rule of Thumb: If you find yourself in the situation above, where your guess turns out to be a multiple of a solution to the homogeneous equation, try multiplying by your guess for $Y(t)$ by t. If this does not work, try multiplying by t^2.

Let $P_n(t) = a_0t^n + a_1t^{n-1} + \cdots + a_n$ be a polynomial of degree n. Here are some guidelines for choosing $Y(t)$.

If $g(t)$ has the form	you should try, for $Y(t)$,
$P_n(t)$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)$
$P_n(t)e^{\alpha t}$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}$
$P_n(t)e^{\alpha t}\sin\beta t$ or $P_n(t)e^{\alpha t}\cos\beta t$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}\cos\beta t$ $+\,t^s(B_0t^n + B_1t^{n-1} + \cdots + B_n)e^{\alpha t}\sin\beta t$

Here s is the lowest power of t that will assure that the resulting guess for $Y(t)$ is not a solution to the corresponding homogeneous equation.