More on the Differential Equation

$$ay'' + by' + cy = 0 :$$

Repeated Roots.

We now know how to handle the differential equation

$$ay'' + by' + cy = 0$$

in the cases where

$$b^2 - 4ac \neq 0.$$

Recall that we considered separately the cases

$$b^2 - 4ac > 0 \qquad \text{or} \qquad b^2 - 4ac < 0.$$

What do we do when

$$b^2 - 4ac = 0 \quad ?$$

In this case, the roots of the characteristic equation

$$ar^2 + br + c = 0$$

are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{0}}{2a} = -\frac{b}{2a}.$$

How do we solve the differential equation

$$ay'' + by' + cy = 0$$

when the two roots r_1, r_2, of the characteristic equation

$$ar^2 + br + c = 0$$

are the same number? That is, what do we do when

$$r_1 = r_2 = -\frac{b}{2a} \quad ?$$

In the case where $b^2 - 4ac = 0$, the zero

$$r_1 = r_2 = -\frac{b}{2a}$$

gives us only one solution:

$$y_1(t) = e^{-bt/(2a)}.$$

But the general solution to the differential equation

$$ay'' + by' + cy = 0 \tag{1}$$

is of the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t), \tag{2}$$

where $y_2(t)$ is a solution to (1) linearly independent to $y_1(t)$.

How do we find a general solution of the form (2)?

Answer: We guess that $y(t)$ is the product of $y_1(t)$ and another function. That is, we guess that

$$y(t) = v(t)y_1(t) = v(t)e^{-bt/(2a)},$$

where $v(t)$ is a function that we need to figure out.

If we assume that $y(t)$ has this form, then, by the product rule,

$$y'(t) = v'(t)e^{-bt/(2a)} - \frac{b}{2a}v(t)e^{-bt/(2a)}. \qquad (3)$$

Applying the product rule to (3) and simplifying, we get

$$y''(t) = v''(t)e^{-bt/(2a)} - \frac{b}{a}v'(t)e^{-bt/(2a)} + \frac{b^2}{4a^2}v(t)e^{-bt/2a}. \qquad (4)$$

Then, substituting (3) and (4) into our original DE,

$$ay'' + by' + cy = 0,$$

we get

$$a\left[v''(t)e^{-bt/(2a)} - \frac{b}{a}v'(t)e^{-bt/(2a)} + \frac{b^2}{4a^2}v(t)e^{-bt/2a}\right]$$
$$+ b\left[v'(t)e^{-bt/(2a)} - \frac{b}{2a}v(t)e^{-bt/(2a)}\right]$$
$$+ c\left[v(t)e^{-bt/(2a)}\right] = 0.$$

We now multiply both sides of this equation by $e^{bt/(2a)}$.

Next, we collect the terms involving $v(t), v'(t)$, and $v(t)$:

$$av''(t) + (-b+b)v'(t) + \left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right) v(t) = 0. \tag{5}$$

Equation (5), in turn, implies that

$$av''(t) + \frac{b^2 - 2b^2 + 4ac}{4a} v(t) = 0. \tag{6}$$

Since $b^2 - 4ac = 0$, we know that $-b^2 + 4ac = 0$, so that

$$\frac{b^2 - 2b^2 + 4ac}{4a} = \frac{-b^2 + 4ac}{4a} = \frac{0}{4a} = 0.$$

From equation (6), we obtain

$$av''(t) = 0,$$

and, hence,

$$v''(t) = 0.$$

Integrating once, we get

$$v'(t) = c_1, \qquad \text{some } c_1 \in \mathbb{R}.$$

Integrating again, we see that

$$v(t) = c_1 t + c_2, \qquad \text{some } c_1, c_2 \in \mathbb{R}.$$

Therefore, since $y_1(t) = e^{-bt/(2a)}$,

$$y(t) = v(t)y_1(t) = (c_1 t + c_2)y_1(t) = c_1 t e^{-bt/(2a)} + c_2 e^{-bt/(2a)}.$$

Therefore, $y(t)$ is a linear combination of the functions

$$y_1(t) = e^{-bt/(2a)} \qquad \text{and} \qquad y_2(t) = te^{-bt/(2a)}.$$

In this case, we can compute the Wronskian of these two functions:

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-bt/(2a)} & te^{-bt/(2a)} \\ -\frac{b}{2a}e^{-bt/(2a)} & \left(1 - \frac{bt}{2a}\right) e^{-bt/(2a)} \end{vmatrix} = e^{-bt/(2a)}. \tag{7}$$

Since the Wronskian is never zero, we know that

$$y_1(t) = e^{-bt/(2a)} \qquad \text{and} \qquad y_2(t) = te^{-bt/(2a)}$$

are linearly independent on the real numbers, and therefore form a fundamental set of solutions to the original differential equation.

We have therefore proven the following useful theorem.

Theorem. *In the case*

$$b^2 - 4ac = 0,$$

the general solution to the differential equation

$$ay'' + by' + cy = 0$$

is given by

$$y(t) = c_1 e^{-bt/(2a)} + c_2 te^{-bt/(2a)}.$$

Example. *Find the general solution to the differential equation*

$$y'' - 6y' + 9y = 0.$$

Here

$$a = 1, \qquad b = -6, \qquad c = 9,$$

so

$$b^2 - 4ac = 6^2 - 4(1)(9) = 36 - 36 = 0.$$

The characteristic equation

$$r^2 - 6r + 9 = 0$$

has the repeated roots

$$r_1 = r_2 = -\frac{b}{2a} = -\frac{-6}{2} = 3.$$

By the theorem on the last page, the general solution to the differential equation is

$$y(t) = c_1 e^{3t} + c_2 t e^{3t}.$$

We can now combine the three cases on $b^2 - 4ac$ in the following theorem.

Theorem. *Consider the differential equation*

$$ay'' + by' + cy = 0. \tag{8}$$

Let r_1 and r_2 be the roots of the correspondong characteristic equation

$$ar^2 + br + c = 0.$$

Then one of the three following cases occurs:

1. *If $b^2 - 4ac > 0$, then r_1 and r_2 are real, and the general solution to the differential equation (8) is*

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

2. *If $b^2 - 4ac < 0$, then r_1 and r_2 are complex numbers, say $\lambda \pm \mu i$. In this case the general solution to (8) is*

$$y(t) = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t.$$

3. *If $b^2 - 4ac = 0$, then $r_1 = r_2 = -\frac{b}{2a}$, and the general solution to (8) is*

$$y(t) = c_1 e^{-bt/(2a)} + c_2 t e^{-bt/(2a)}.$$

Reduction of Order
(Or how to get a new solution from an old one)

To solve

$$ay'' + by' + cy = 0$$

in the case where $b^2 - 4ac = 0$, we made a lucky guess based upon a **known solution**.

We can extend this to a general procedure that will give us a solution to the equation

$$y'' + p(t)y' + q(t)y = 0$$

from a single **known solution.**

Reduction or Order Algorithm: If $y_1(t)$ is a known solution to the differential equation

$$y'' + p(t)y' + q(t)y = 0,$$

then *you might* be able to find another solution by trying

$$y(t) = v(t)y_1(t),$$

and then using the differential equation to determine what the function $v(t)$ is.

Example. *Given that* $y_1(t) = t^{-1}$ *is a solution to the differential equation*

$$2t^2y'' + 3ty' - y = 0, \qquad t > 0,$$

find the second linearly independent solution.

Let's try the reduction of order algorithm, setting

$$y(t) = v(t)t^{-1}.$$

Then, by the product rule,

$$\begin{aligned} y'(t) &= v'(t)t^{-1} - v(t)t^{-2}, \\ y''(t) &= v''(t)t^{-1} - 2v'(t)t^{-2} + 2v(t)t^{-3}. \end{aligned}$$

Let's now plug this back into the original differential equation:

$$\begin{aligned} 2t^2\Big(v''(t)t^{-1} - 2v'(t)t^{-2} + 2v(t)t^{-3}\Big) \\ + 3t\Big(v'(t)t^{-1} - v(t)t^{-2}\Big) - v(t)t^{-1} = 0. \end{aligned}$$

This simplifies to

$$2tv''(t) - v'(t) = 0. \tag{9}$$

We now think of (9) as a first order equation in the function $w(t) = v'(t)$. We can then write (9) as

$$2tw'(t) - w(t) = 0 \qquad \text{or} \qquad 2t\frac{dw}{dt} = w.$$

We can solve this separable equation by writing the last equation in the form

$$\frac{dw}{w} = \frac{dt}{2t}.$$

Integrating both sides gives

$$\ln w = \frac{1}{2}\ln t + K = \ln t^{1/2} + K, \qquad w > 0, \qquad K \in \mathbb{R}.$$

Thus, by exponentiating, we find that

$$w(t) = ce^{\ln t^{1/2}} = ct^{1/2}, \qquad c = e^K.$$

(Our assumption that $w(t) = ct^{1/2} > 0$ is satisfied since $t > 0$ by assumption.)

Therefore,

$$v'(t) = w(t) = ct^{1/2}.$$

Integrating this equation, we find that

$$v(t) = \frac{2}{3}ct^{3/2} + k.$$

Since $y_1(t) = t^{-1}$,

$$y(t) = v(t)y_1(t) = c_1 t^{1/2} + c_2 t^{-1}, \qquad c_1 = \frac{2}{3}c,\ c_2 = k.$$

We can drop the last term since it contains the solution we started with.

We see that $y_2(t) = t^{1/2}$ is an independent solution since

$$W(y_1, y_2)(t) = \begin{vmatrix} t^{-1} & t^{1/2} \\ -t^{-2} & \frac{1}{2}t^{-3/2} \end{vmatrix} = \frac{1}{2}t^{-5/2} + t^{-3/2}.$$

In particular

$$W(y_1, y_2)(1) = 2 \neq 0.$$