Math 165 - Homework Assignment 1 Solution

Name: ______________________________

Write your solutions to these problems on a **separate** sheet of paper. Show **all** work to receive full credit for each problem. Turn in complete, legible, organized, and logically sound solutions and arguments. Give exact answers, not decimal approximations. This assignment is worth 10 points and is due **Thursday, January 24** in class. I will randomly select 2 of the following 5 problems to grade.

1. Prove, using the $\epsilon - \delta$ definition of a limit, that $\lim_{x \to 2} 3x + 1 = 7$

Solution: Let $\epsilon > 0$. Choose $\delta = \dfrac{\epsilon}{3} > 0$. Then if $0 < |x-2| < \delta$, $|3x+1-7| = |3x-6| = 3|x-2| < 3\delta = \epsilon$. So $\lim_{x \to 2} 3x+1 = 7$ by definition.

2. Prove, using the $\epsilon - \delta$ definition of a limit, that $\lim_{x \to 1} \sqrt{2x} = \sqrt{2}$

Solution: Let $\epsilon > 0$. Choose $\delta = \dfrac{\epsilon\sqrt{2}}{2}$. Then if $0 < |x-1| < \delta$, $|\sqrt{2x} - \sqrt{2}| = \left|\dfrac{2x-2}{\sqrt{2x}+\sqrt{2}}\right| \leq \left|\dfrac{2|x-1|}{\sqrt{2}}\right| < \dfrac{2\delta}{\sqrt{2}} = \epsilon$. So $\lim_{x \to 1} \sqrt{2x} = \sqrt{2}$ by definition.

3. Find an example of two functions f and g and a real number c such that $\lim_{x \to c} f(x)g(x)$ exists but $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ do not exist.

Solution: For example, let $c = 0$ and define

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ 0 & \text{if } x > 0 \end{cases}$$

Then clearly $f(x)g(x) = 0$ for all x and hence $\lim_{x \to 0} f(x)g(x) = 0$ while neither $\lim_{x \to 0} f(x)$ nor $\lim_{x \to 0} g(x)$ exist.

4. Let $f(x) = x^3 \cos\left(\dfrac{1}{x^7}\right)$ for $x \neq 0$. Prove that $\lim_{x \to 0} f(x) = 0$. [Hint: Compute the left and right-sided limits separately.]

Math 165 - Homework Assignment 1 Solution

Solution: Recall that for all θ, $-1 \le \cos(\theta) \le 1$ and hence $-1 \le \cos\left(\frac{1}{x^7}\right) \le 1$ for all $x \neq 0$.

For $x > 0$, this implies $-x^3 \le x^3 \cos\left(\frac{1}{x^7}\right) \le x^3$ and hence by the Squeeze Theorem, $\lim_{x \to 0^+} f(x) = 0$.

For $x < 0$, it implies $-x^3 \ge x^3 \cos\left(\frac{1}{x^7}\right) \ge x^3$ and hence by the Squeeze Theorem, $\lim_{x \to 0^-} f(x) = 0$.

Thus $\lim_{x \to 0} f(x) = 0$.

5. Let $h(x) = x - [[x]]$. Evaluate the following limits or show they do not exist.

   (a) $\lim_{x \to 2^+} h(x)$

   Solution: $\lim_{x \to 2^+} [[x]] = 2$ and $\lim_{x \to 2^+} x = 2$, so $\lim_{x \to 2^+} x - [[x]] = 2 - 2 = 0$.

   (b) $\lim_{x \to 7} h(x)$

   Solution: $\lim_{x \to 7^+} x - [[x]] = 0$ but $\lim_{x \to 7^-} x - [[x]] = 7 - 6 = 1$ so the limit $\lim_{x \to 7} h(x)$ does not exist.