Math 515
Real Analysis
Problem Set 8
You may consult with other human beings on these problems.
Due date: November 28, 2005

Each problem is worth 10 points unless otherwise stated.

1. By an algebra of subsets of a set X we mean a collection of subsets $\mathfrak{M}$ of X with the following properties: (i) $\emptyset \in \mathfrak{M}$, (ii), if $A \in \mathfrak{M}$ then $A^c \in \mathfrak{M}$ and finite unions of sets in $\mathfrak{M}$ are in $\mathfrak{M}$. Give an example of an algebra of sets that is not a σ-algebra. Give an example of a set function on a σ-algebra which is finitely additive but not countably additive.

2. Show that if a σ-algebra is countable, then it is finite. Hint: Assume the contrary. If $\mathfrak{M}$ is the sigma algebra and X is the underlying set, X must be infinite. Now for $x \in X$, define $A_x = \cap A \in \mathfrak{M} | x \in A$. Such a set is called an atom.

3. Show that if A_i are measurable subsets of a measure space X and μ is the measure, then

$$\mu\left(\cup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

4. Show that if $\mu(A \Delta B) = 0$ then $\mu(A) = \mu(B)$ ("Δ" denotes symmetric difference.) If μ is complete, A is measurable and $\mu(A \Delta B) = 0$, then B is measurable.

5. 20 points. A measure space $\langle X, \mathfrak{M}, \mu\rangle$ is said to be σ-finite if there is a sequence of measurable sets $\{A_i\}_{i=1}^{\infty}$ with $\mu(A_i) < \infty$ and $X = \cup_{i=1}^{\infty} A_i$.

(i) Show that if $X = (-\infty, \infty)$ and $\mathfrak{M} =$ Lebesgue measurable subsets, $\mu =$ Lebesgue measure, then $\langle X, \mathfrak{M}, \mu\rangle$ is σ-finite.

(ii) Give an example of a measure space which is not σ-finite.

(iii) Show that in a σ-finite measure space with $\mu(X) = \infty$, for every $M > 0$ there is a measurable set with **finite** measure $\mu(A) > M$.

6. Suppose μ is a positive measure on $\mathbf{X}$, $f : \mathbf{X} \longmapsto [0, \infty]$ is measurable, $\int_{\mathbf{X}} f \, d\mu = c$, $0 < c < \infty$ and $\alpha > 0$. Find

$$\lim_{n\to\infty} \int_{\mathbf{X}} n \log\left(1 + (f/n)^{\alpha}\right) d\mu$$

(Hint: Use the dominated convergence theorem for $\alpha > 1$ and Fatou's lemma for $0 < \alpha < 1$.)

More problems on next page!!

7. Prove the uniform integrability property: If $f \in L^1(\mu)$ then for every $\epsilon > 0$ there exists $\delta > 0$ such that $\int_E f\, d\mu < \epsilon$ whenever $\mu(E) < \delta$. (Hint: Approximate f by simple functions.)

8. Explain what is wrong with the following argument: The function $f(x) = \dfrac{1}{1+x^2}$ is in $L^1(0,\infty)$ with respect to Lebesgue measure. Define

$$f_n(x) = ne^{-n^2x^2} f(x)$$

for $n = 1, 2, ...$ This sequence converges to zero a.e. Therefore by the Dominated Convergence Theorem (DCT)

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\, dx = 0.$$

On the other hand, after the change of variable $y = nx$ we have

$$\begin{aligned}\lim_{n\to\infty} \int_0^\infty f_n(x)\, dx &= \lim_{n\to\infty} \int_0^\infty \frac{n^2}{n^2+y^2} e^{-y^2}\, dy \\ &= \int_0^\infty e^{-y^2}\, dy \quad \text{(by DCT again. Why?)} \\ &= \frac{\sqrt{\pi}}{2}.\end{aligned}$$

Therefore $\pi = 0$????