

Math 515  
Real Analysis  
Problem Set 5

You may **not** consult with other human beings on these problems

Due date: October 24, 2005

Each problem is worth 10 points unless otherwise stated.

1. 20 points. Let  $(X, \rho)$  be a metric space. Is  $(X, \rho^2)$  a metric space? If  $(X, \rho_1), (X, \rho_2)$  are metric spaces, is the same true for each of the following "spaces":  $(X, \rho_1 + \rho_2), (X, \max(\rho_1, \rho_2))$  and  $(X, \min(\rho_1, \rho_2))$ ? Support your answers with proofs or counter examples.

2. Suppose  $\rho : X \times X \rightarrow [0, \infty)$  is such that  $\rho(x, y) = 0$  if and only if  $x = y$ . Suppose that  $\rho(x_1, x_2) \leq \rho(x_1, x_3) + \rho(x_2, x_3)$  for all  $x_1, x_2, x_3 \in X$ . Show that  $\rho$  is a metric.

3. Suppose  $(X, \rho)$  is a metric space. Prove that  $(X, \rho_1)$  is also a metric space where

$$\rho_1(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)}.$$

4. Show that the statement that arbitrary intersections of open sets are open in a metric space is equivalent to the statement that all singleton sets (sets consisting of a single point) are open.

5. Show that  $\{x_i | i = 1, 2, \dots\}$  is a bounded set if  $\{x_i\}_{i=1}^{\infty}$  is a Cauchy sequence.

6. Suppose  $(X, \rho)$  is a metric space. If  $\lim_{i \rightarrow \infty} x_i = x$  and  $\lim_{i \rightarrow \infty} y_i = y$  then  $\lim_{i \rightarrow \infty} \rho(x_i, y_i) = \rho(x, y)$ .

7. Suppose  $(X, \rho)$  is a metric space and  $A \subset X$ . If  $x \in \overline{A}$ , there is a sequence of elements of  $A$  which converges to  $x$ .