

Math 515
Real Analysis
Problem Set 4

You may consult with other human beings on these problems
Due date: October 10, 2005

Each graded problem is worth 10 points.

1. Show that the dominated convergence theorem follows from Fatou's theorem. (Hint: consider $g_n = 2F - |f_n - f|$.)
2. Derive Chebyshev's inequality

$$m(\{x \mid |f(x)| \geq n\}) \leq \frac{\int |f(x)| dm(x)}{n}.$$

3. Suppose that f is integrable on a set E and $f_n \rightarrow f$ a. e. Assume that the $f_n \geq 0$ almost everywhere. Show that $\int |f_n(x) - f(x)| dm(x) \rightarrow 0$ if and only if $\int_E f_n(x) dm(x) \rightarrow \int_E f(x) dm(x)$.
Hint: Apply Fatou's lemma with $g_n = |f| + |f_n| - |f - f_n|$. Show that if the hypothesis that the f_n are nonnegative is omitted, the reverse implication is false. Hint: Consider odd functions.

4. Suppose $a < b$ and $a' < b'$. Let

$$f(x) = x \cdot \begin{cases} a \sin^2\left(\frac{1}{x}\right) + b \cos^2\left(\frac{1}{x}\right) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ a' \sin^2\left(\frac{1}{x}\right) + b' \cos^2\left(\frac{1}{x}\right) & \text{if } x < 0. \end{cases}$$

Find the derivatives $D^+ f(0)$, $D_+ f(0)$, $D^- f(0)$, $D_- f(0)$.

5. Prove the following: If $f(x)$ is continuous on $[a, b]$ and one derivate is nonnegative on the open interval (a, b) , then $f(a) \leq f(b)$.

6. Show that

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous on $[-1, 1]$ but is not of bounded variation there. Is

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

of bounded variation? Why or why not?