MATH 557, Fall 2007, Homework set # 4, Due: September 21,2007

1. Write down the Pickard iteration scheme for $y' = 3y^{2/3}$ with $y(0) = 0$ and an arbitrary starting function $y_0(t) = \varphi(t)$ with $\varphi(0) = 0$. Generate the sequence for each of the following two choices of $\varphi(\cdot)$: (a). $\varphi(t) \equiv 0$. (b). For $c > 0$, $\varphi(t) = 0$ if $t \leq c$ and $\varphi(t) = t - c$ if $t \geq c$. What are the limits of each such sequence?

2. Find out how to obtain a numerical solution of a system of ODEs in MATLAB. (Suggestion: give the command `help ode45`.) Do it for the initial value problem

$$\begin{aligned} y_1' &= \sin(x^2 + y_2^2) \qquad && y_1(1) = 1 \\ y_2' &= \frac{y_1 + y_3}{x} && y_2(1) = 0 \\ y_3' &= y_2 - y_1^3 && y_3(1) = 2 \end{aligned}$$

and plot the results for $1 \leq x \leq 2$.

3. Prove the Osgood uniqueness theorem: Let h be a continuous, increasing function on some interval $[0, a]$, $h(t) > 0$ for $0 < t \leq a$, such that

$$\int_0^a \frac{1}{h(t)}\, dt = \infty \tag{1}$$

If $f(x, y)$ is continuous and satisfies the condition

$$|f(x, y) - f(x, \hat{y})| \leq h(|y - \hat{y}|) \tag{2}$$

for all $x, y, \hat{y}$, show that the solution of

$$y' = f(x, y) \qquad y(\xi) = \eta$$

is unique. (Suggestions: Suppose y_1, y_2 are both solutions and let

$$z(x) = |y_1(x) - y_2(x)|$$

Show that $z(x) \leq \int_\xi^x h(z(t))\, dt$ and so $w'(x) \leq h(w(x))$ where $w(x) = \int_\xi^x h(z(t))\, dt$. Show that w must be zero.) Note that $h(t) = Lt$ satisfies all the requred conditions, in which case the condition (2) on f is the usual Lipschitz condition. Find an example of a function $f(x, y)$ satisfying (1),(2) which is not Lipschitz continuous.