

Math 515  
Real Analysis  
Problem Set 8

You may consult with other human beings on these problems

Due date: December 17, 2008

Each problem is worth 10 points unless otherwise stated.

1. Prove that a closed set is nowhere dense if and only if it contains no open set. Prove that  $E$  is nowhere dense if and only if for every open set  $O$  there is a nonempty ball in  $O - E$ .
2. Prove that if  $A, B$  are closed disjoint subsets of a metric space and one of them, (say  $A$ ), is compact, then  $\rho(A, B) > 0$ .
3. If  $A, B$  are nowhere dense in  $X$ , then so is  $A \cup B$ . If  $A$  is nowhere dense in  $X$  and  $B$  is nowhere dense in  $Y$ , then  $A \times B$  is nowhere dense in  $X \times Y$ .
4. If  $X, Y$  are first category, so is  $X \times Y$ . Prove that if  $\{E_n\}$  is a sequence of first category sets, so is their union.
5. Definition: A point  $x \in X$  is isolated if it is not a limit point of  $X$ . Prove that a space of the first category has no isolated points. Prove that a complete, countable metric space must have isolated points.
6. Show that on  $[0, 1]$ , there is a nowhere dense closed set having Lebesgue measure  $1 - 1/n$ . Construct a first category set in  $[0, 1]$  with Lebesgue measure one.
7. 25 pts. Work problem 38, p 161 of the text.