Math 515
Real Analysis
Problem Set 5
Due date: October 20, 2008
Consultation with other individuals is not permitted for this problem set.
Each graded problem is worth 10 points.

1. Show that the Lebesgue Dominated Convergence Theorem is false for Riemann Integrals.

2. Use Egoroff's Theorem (even if you couldn't prove it) to prove the Lebesgue Dominated Convergence Theorem for measureable functions (which need not be bounded).

3. Show that $f(x) \equiv x^{-a}$ for $0 < x \leq 1$ is integrable in the sense of Lebesgue if and only if $a < 1$ and evaluate the integral in the latter case.

4. Let $f(x) = x^2 \sin\left(\frac{1}{x^2}\right)$ and $h(x) = f'(x)$ on $0 < x \leq 1$. (Here we mean the usual calculus notion of a derivative.) Show that $h(x)$ is not Lebesgue integrable but that the improper Reimann integral of $h(x)$ in the sense that

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^1 h(x)\, dx$$

exists.

5. Suppose $f \geq 0, g \geq 0$. If

$$f_n(x) \equiv \begin{cases} n & \text{if } \ f(x) > n \\ f(x) & \text{if } \ f(x) \leq n. \end{cases}$$

Show that

$$(f+g)_n \leq f_n + g_n \leq (f+g)_{2n}.$$

6. Let C be the Cantor set in $[0,1]$. Let

$$f(x) = \begin{cases} 0 & \text{if } x \in \text{C} \\ p & \text{if } x \in \text{a complementary interval of length } 3^{-p}. \end{cases}$$

Prove that

$$\int_0^1 f(x)\, dm(x) = 3.$$

7. Show that f is absolutely integrable on a set E if and only if

$$\sum_{n=-\infty}^{+\infty} |n|\, m(E \cap f^{-1}([n-1, n))) < +\infty.$$