Private Debt with Pervasive Risk of Default

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Abstract

This paper studies the effects of private debt on risk sharing and welfare, where individual residents are assumed to have access to both international and domestic asset markets. Like Jeske (2006), the assumption is that domestic residents cannot commit to repay their debts across borders. Unlike previous literature, the novel feature in this paper is to bring limited commitment to debt contracts within borders. The marginal rate of substitution (henceforth, MRS) is no longer necessarily equalized among all residents in any one country. The pervasive risk of default creates heterogeneity in MRS for countries, as a whole, constrained in the international asset market. A constrained country's domestic interest rate is equal to the reciprocal of the lowest MRS within that country. However, non-constrained countries still have equalized MRS, which determines the international interest rate. A wider gap between international and domestic financing cost emerges. This leads to harsher punishment for international debt defaulters; hence, allows more international risk sharing. Although limited commitment within borders hinders domestic risk sharing from reaching an even higher standard, it has no negative impact on the original level in Jeske's setup. As a result, this study's modification can support higher aggregate welfare. This paper shows how this improvement depends upon the interaction between the endogenous borrowing constraints in international and domestic asset markets. This study explores also the role of borrowing through intermediaries, endowment structure, and different specifications on default penalty.

Keywords: Default risk, private debt, limited commitment.

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1 Introduction

In the presence of limited commitments, regardless of complete financial markets, international loans are made available only to the extent their repayments can be enforced by the threat of imposing penalties, such as reversion to autarky, to the debtor country. This commiment problem results in limited risk sharing among countries, and the risk-sharing size is determined by the specification of outside options. Jeske (2006) predicts a centralized arrangement, where only a government borrows internationally and redistributes domestically, allows more international risk sharing and higher aggregate welfare than a decentralized arrangement, where individuals have access to capital markets. An intuitive proof is as follows. One can think of the decentralized arrangement as a centralized setup that only this imaginary government assumes it can ignore the resource constraint in autarky and continue to borrow at domestic interest rates like individuals—a better outside option than pure autarky in the original centralized arrangement. A higher post-default value leads to tighter participation constraints in the international asset market and, accordingly, a smaller capital inflow. Perfect enforcement of domestic debt is the crucial assumption here.

This paper relaxes Jeske's assumption and introduces a new method of deviation from risk sharing agreements—domestic debt default. While international debt defaulters are excluded from the international asset market, but retain access to the domestic asset market, domestic debt defaulters are punished harsher by exclusion from all asset markets. This specific discrimination against foreign creditors seems more realistic than the previous infinite discrimination, where domestic creditors are fully protected. Limited commitment on domestic debt has opposite effects on aggregate welfare after opening to the outside world. Pervasive risk of default may hamper domestic risk sharing and worsen welfare; whereas, it can raise the volume of international capital flow, thus improving welfare. The logic behind the latter statement is based on the following observation. Because of the participation constraints in the domestic asset market, international debt defaulter's scheme of using other non-defaulted domestic residents as intermediaries to re-access international asset market is restricted. In contrast, the international debt defaulter in Jeske's setup can trade *freely* and indirectly after the default. Then, the question is which effect dominates the direction of welfare movement. This paper will prove that when residents are heterogeneous and there exists someone, who will be on the edge of reneging on domestic debt, the positive effect allows them to enjoy welfare gains from an increase in foreign capital inflow; meanwhile, the welfare levels for the others remain unchanged. Although the negative effect hinders aggregate welfare reaching an even higher standard when compared with the centralized arrangement, it does not function when compared with Jeske's decentralized arrangement. Therefore, more international risk sharing and higher aggregate welfare can be supported. The policy implication is that perfect domestic risk sharing and infinite discrimination against foreign creditors, together, lower the level of international risk sharing, if pervasive risk of default is the norm. There is a rationale for government in decentralized countries to sacrifice domestic enforcement for more international risk sharing.

Now that aggregate welfare can be improved by not enforcing domestic debt, does the government still want to control private capital and achieve the centralized arrangement? The answer depends upon the trade-off that centralization wins by gains from domestic exchange, but may lose by losses from international exchange. Using a numerical example, it will be demonstrated that capital control is superior to this model if the endowment structure is such that income fluctuation across countries is large relative to income fluctuation across types within one country. Otherwise, limited commitment on domestic debt will lead to greater welfare increment. Using this idea, a similar critical condition will be identified in the formal model under which the government is willing to give up private debt prohibition.

The limited commitment problem within a counry's border also overturns the equilibrium domestic bond pricing rule for countries that participate constrained in international asset market. In closed economy models, the domestic interest rate is determined by the highest MRS in that economy to ensure repayment.[1] In open economy models with perfect enforcement on domestic debt, the international interest rate is determined by the highest MRS in all countries across the world, whereas domestic interest rate in any country is determined by an equalized MRS.[2] In this model, the international interest rate is, as usual, determined by the highest MRS. However, the domestic interest rate in any constrained country is equal to the reciprocal of the lowest MRS among all of its residents. This result is consistent with the assumption that international debt defaulters are penalized more severely. When international debt defaulters come seeking help in the domestic asset market, they face a wider gap between domestic and international financing costs, since the domestic bond price is lower than ever. This bad situation raises the international borrowing quota for these potential defaulters and rewards them at a higher welfare level.

Of critical importance in this literature is what defaulters might be entitled. Several closely related works replace complete exclusion[3] with partial exclusion, under which defaulters retain some access to asset markets or have alternative ways to smooth consumption. This induces international risk sharing to diminish

[1] Bond price is negatively related to the corresponding interest rate. When price is determined by the highest MRS, the interest rate is the lowest possible. See, for example, Alvarez and Jermann (2000, 2001) and Azariadis and Lambertini (2002).

[2] See, for example, Jeske (2006) and Wright (2006).

[3] See, for example, Kehoe and Levine (1993), Kocherlakota (1996), Alvarez and Jermann (2000) and Kehoe and Perri (2002, 2004)

further in size, since life after a default is less painful than it would be otherwise. Partial exclusion arises if defaulters can reenter the international asset market indirectly through intermediaries as in Jeske (2006). Wright (2006) builds on Jeske's model and argues that international borrowing subsidies can also lead to constrained efficient allocations, instead of Jeske's radical way of centralization. A defaulter continuing to have access to international savings causes partial exclusion as well. Bulow and Rogoff (1989) first use this idea and prove that borrowing cannot be supported in a small open economy that takes the international interest rate as given (partial equilibrium). Hellwig and Lorenzoni (2007) carry their work further to a multi-country (general equilibrium) setup, where they show international risk sharing can exist with low interest rates. Then, Wright (2006) establishes an equivalence between the above two modeling methods, if the extra dimension of heterogeneity among residents in Jeske's model is accommodated. Reduced penalty can be due to other internal opportunities as well. For instance, Kehoe and Perri (2002, 2004) study international risk sharing in a real business cycle model with productivity shock, where the autarky value depends upon the quantity of capital the country has accumulated to default. In their paper, defaulters continue to produce and employ capital in autarky, but they are not allowed to buy or sell capital and other financial assets.

Broner and Ventura (2009) assume that countries cannot discriminate against foreign creditors. Thus, international risk sharing is obtained even in the absence of default penalty. Unlike this paper's model, where residents make default decisions (decentralized arrangement) and the government only decides whether or not to enforce domestic debt, the government in its setup makes default decisions on behalf of all residents (centralization), and chooses endogenously whether to enforce all debt contracts or none. Broner and Ventura (2009) show a decrease of trade barriers in goods market facilitates international trade and raises the cost of enforcement. As a result, its government may favor enforcing none after globalization—preventing large amount of capital outflow at the cost of hampering domestic trade. Consistent with their findings, the government in this study makes the same choice of not enforcing domestic debt in an open world economy for similar reasons.

The remainder of this paper is organized as follows. Section 2 presents the model and derives equilibrium results. Section 3 compares aggregate welfare levels and, thus, generates implications. Section 4 introduces a crude example aimed at the essence of the results. Section 5 concludes, followed by a technical appendix containing all proofs.

2 The Model

The model considers a world that consists of a finite number of countries denoted as $m \in \{1, 2, ..., M\}$ and each country, m, is populated by N types of residents with a continuum in each type $n \in \{1, 2, ..., N\}$.[4] Residents live forever, so that time is infinite and discrete, as denoted by $t = 0, 1, 2, ..., \infty$. Information about current and future endowments is indexed by the state $\theta_t \in \Theta$. History is summarized in $\theta^t \equiv \{\theta^0, \theta_1, \theta_2, ..., \theta_t\} \in \Theta^t$ with θ^0 given. Transition probability from history θ^t to the next period's state θ_{t+1} is known as $\pi(\theta_{t+1}|\theta^t)$ with θ^t given. $\pi(\theta^t)$ is the unconditional probability of observing history θ^t and $\pi(\theta^r|\theta^t)$ is the probability of observing history θ^r conditional on having been in θ^t. There is only one non-storable-consumption goods, which can be exchanged within and across borders. This is denoted by $e_n^m(\theta^t)$, the endowment of type n in country m after history θ^t, and by $c_n^m(\theta^t)$, the corresponding consumption. There are M domestic bonds—one for each country, m, and only one international bond traded across the world. Let $b_n^m(\theta^t, \theta_{t+1})$ and $f_n^m(\theta^t, \theta_{t+1})$, respectively, be the amounts of domestic and international state-contingent securities held by residents of type n living in country m, purchased at θ^t and for payment next period in state θ_{t+1}; $p^m(\theta^t, \theta_{t+1})$, and $q(\theta^t, \theta_{t+1})$ are their respective prices.

For all n and m, use $\beta \in (0, 1)$ as the discount factor and denote $U(\cdot)$, as the period utility function, which is strictly increasing, strictly concave, and twice continuously differentiable. Type n residents in country m have preference,

$$\sum_{r=t}^{\infty} \beta^{r-t} \sum_{\theta^r|\theta^t} \pi(\theta^r|\theta^t) U(c_n^m(\theta^r)),$$

after θ^t with $t \in [0, \infty)$.

Pervasive risk of default means that debt contracts between any two parties are not enforced. Border is still important here because the default within the border leads to a harsher penalty than the penalty for default across the border. Domestic debt defaulters are denied from both international and domestic asset markets. Their post-default value at θ^r is

$$A_n^m(\theta^r) \equiv \sum_{s=r}^{\infty} \beta^{s-r} \sum_{\theta^s|\theta^r} \pi(\theta^s|\theta^r) U(e_n^m(\theta^s)), \tag{RA}$$

[4] Unlike Jeske, this paper assumes that in any country m, the mass of type n residents λ_n^m is normalized to 1 for all $n \in \{1, 2, ..., N\}$. Note, individual endowment in Jeske's model only depends on type. That is, although living in different countries, the same types receive the same endowment each period. As a result, using $\lambda_n^m = 1$ for all n and m in his model would imply that countries are identical ex-ante. Thus, no role exists for international risk sharing. However, individual endowment in this paper varies with both type and country, which will be clear after history and endowment structure. Assuming $\lambda_n^m = 1$ simplifies the notation, but still justifies international risk sharing.

called the **Resident's Autarky,** hereafter. International debt defaulters can still trade international bonds indirectly, through borrowing from other non-defaulted domestic residents in the same country. Refer to this situation as the **Resident's International Autarky**,[5] which offers the following post-default value at θ^t.

$$V_n^m(\theta^t, b_n^m(\theta^t)) \equiv \max_{\{c_n^m(\theta^r), b_n^m(\theta^r,\theta_{r+1})\}_{r\in[t,\infty)}} \sum_{r=t}^{\infty} \beta^{r-t} \sum_{\theta^r|\theta^t} \pi(\theta^r|\theta^t) U(c_n^m(\theta^r)), \tag{RIA}$$

subject to the budget constraint,

$$e_n^m(\theta^r) + b_n^m(\theta^r) \geqslant c_n^m(\theta^r) + \sum_{\theta_{r+1}} p^m(\theta^r, \theta_{r+1}) b_n^m(\theta^r, \theta_{r+1}), \tag{1}$$

the participation constraint in domestic asset market,

$$\sum_{s=r}^{\infty} \beta^{s-r} \sum_{\theta^s|\theta^r} \pi(\theta^s|\theta^r) U(c_n^m(\theta^s)) \geqslant A_n^m(\theta^r), \tag{2}$$

and the no-Ponzi condition,

$$b_n^m(\theta^r, \theta_{r+1}) \geqslant -\overline{B},$$

with $b_n^m(\theta^t)$ and $\{p^m(\theta^r, \theta_{r+1})\}_{r\in[t,\infty)}$ given, for all histories θ^r and states (θ^r, θ_{r+1}) with $r \in [t, \infty)$.

$b_n^m(\theta^t)$ denotes the domestic bond holding, inherited before entering period t. $\overline{B} > 0$ is sufficient large, such that the no-Ponzi condition never binds in equilibrium, which ensures compactness of the budget set. For the problem to be interesting, assume for some n, m, and θ^t, there exist future histories θ^r, where domestic participation constraints (2) are binding.

First order condition with respect to $c_n^m(\theta^r)$ is

$$\beta^{r-t}\pi(\theta^r|\theta^t)U'(c_n^{m,D}(\theta^r)) - \lambda_n^m(\theta^r) + \sum_{s=t}^{r} \nu_n^m(\theta^s)\beta^{r-s} \sum_{\theta^r|\theta^s} \pi(\theta^r|\theta^s)U'(c_n^{m,D}(\theta^r)) = 0, \tag{3}$$

where $\lambda_n^m(\theta^r)$ and $\nu_n^m(\theta^r)$ denote respectively the Lagrangian multipliers on the budget constraint (1) and domestic participation constraint (2) if θ^r occurs. Let $\left\{c_n^{m,D}(\theta^r)\right\}_{r\in[t,\infty)}$ be the optimal consumption path to problem (RIA) with initial history θ^t.

To this point, the outside options for both domestic and international debt defaulters have been defined. Next, the **Resident's Problem** is noted before any default. Residents choose sequences for consumption, and for both domestic and international bonds to maximize lifetime preference at period 0,

$$\max_{\{c_n^m(\theta^t), b_n^m(\theta^t,\theta_{t+1}), f_n^m(\theta^t,\theta_{t+1})\}_{t\in[0,\infty)}} \sum_{t=0}^{\infty} \beta^t \sum_{\theta^t} \pi(\theta^t) U(c_n^m(\theta^t)), \tag{RP}$$

[5] Since individual resident's influences are miniscule, relative to the market, an individual defaulter on the international debt does so at θ^t, by assuming the sequence of all future domestic bond prices stays unchanged.

subject to the budget constraint,

$$\begin{aligned} & e_n^m(\theta^t) + b_n^m(\theta^t) + f_n^m(\theta^t) \\ \geqslant \; & c_n^m(\theta^t) + \sum_{\theta_{t+1}} p^m(\theta^t, \theta_{t+1}) b_n^m(\theta^t, \theta_{t+1}) + \sum_{\theta_{t+1}} q(\theta^t, \theta_{t+1}) f_n^m(\theta^t, \theta_{t+1}), \end{aligned} \tag{4}$$

the participation constraint in the international asset market,

$$\sum_{r=t}^{\infty} \beta^{r-t} \sum_{\theta^r|\theta^t} \pi(\theta^r|\theta^t) U(c_n^m(\theta^r)) \geqslant V_n^m(\theta^t, b_n^m(\theta^t)), \tag{5}$$

the participation constraint in the domestic asset market,

$$\sum_{r=t}^{\infty} \beta^{r-t} \sum_{\theta^r|\theta^t} \pi(\theta^r|\theta^t) U(c_n^m(\theta^r)) \geqslant A_n^m(\theta^t), \tag{6}$$

and the no-Ponzi conditions,

$$b_n^m(\theta^t, \theta_{t+1}) \geqslant -\overline{B}, f_n^m(\theta^t, \theta_{t+1}) \geqslant -\overline{F},$$

with initial bond holdings,

$$b_n^m(\theta^0) \text{ and } f_n^m(\theta^0),$$

and the price sequence,

$$\{p^m(\theta^t, \theta_{t+1}), q(\theta^t, \theta_{t+1})\}_{t\in[0,\infty)} \text{ given,}$$

for all histories θ^t and states (θ^t, θ_{t+1}) with $t \in [0, \infty)$.

Notice the domestic participation constraint (6) is *redundant* because of

$$V_n^m(\theta^t, b_n^m(\theta^t)) \geqslant A_n^m(\theta^t), \forall \theta^t.$$

Intuitively, no one reneges on domestic debt before a default on an international debt. Hence, limited commitment on domestic debt will only affect the optimal allocation indirectly through $V_n^m(\theta^t, b_n^m(\theta^t))$, the value function of the problem (RIA). The remainder of this section defines and characterizes this equilibrium with international trade of financial assets.

Definition 1 *A* ***Trade Equilibrium*** *is an allocation* $\{c_n^m(\theta^t), b_n^m(\theta^t, \theta_{t+1}), f_n^m(\theta^t, \theta_{t+1})\}_{t\in[0,\infty)}$ *and a price sequence* $\{p^m(\theta^t, \theta_{t+1}), q(\theta^t, \theta_{t+1})\}_{t\in[0,\infty)}$ *such that each type solves its problem (RP) given prices and initial bond holdings, while the resource feasibility,*

$$\sum_{m=1}^{M} \sum_{n=1}^{N} c_n^m(\theta^t) \leq \sum_{m=1}^{M} \sum_{n=1}^{N} e_n^m(\theta^t),$$

the domestic asset market's clearing conditions,

$$\sum_{n=1}^{N} b_n^m(\theta^t, \theta_{t+1}) = 0, \forall m,$$

and the international asset market's clearing condition,

$$\sum_{m=1}^{M}\sum_{n=1}^{N} f_n^m(\theta^t, \theta_{t+1}) = 0,$$

are satisfied for all histories θ^t *and states* $\left(\theta^t, \theta_{t+1}\right)$ *with* $t \in [0, \infty)$*.*

First order conditions are

$$\beta^t \pi(\theta^t) U'(c_n^m(\theta^t)) - \kappa_n^m(\theta^t) + \sum_{s=0}^{t} \mu_n^m(\theta^s)\beta^{t-s} \sum_{\theta^t|\theta^s} \pi(\theta^t|\theta^s) U'(c_n^m(\theta^t)) = 0, \tag{7}$$

$$-p^m(\theta^t, \theta_{t+1})\kappa_n^m(\theta^t) + \kappa_n^m(\theta^t, \theta_{t+1}) - \mu_n^m(\theta^t, \theta_{t+1}) \frac{dV_n^m(\left(\theta^t, \theta_{t+1}\right), b_n^m(\theta^t, \theta_{t+1}))}{db_n^m(\theta^t, \theta_{t+1})} = 0, \tag{8}$$

and

$$-q(\theta^t, \theta_{t+1})\kappa_n^m(\theta^t) + \kappa_n^m(\theta^t, \theta_{t+1}) = 0, \tag{9}$$

where $\kappa_n^m(\theta^t)$ and $\mu_n^m(\theta^t)$ denote respectively the Lagrangian multipliers on the budget constraint (4) and international participation constraint (5) if θ^t occurs. Let $\left\{c_n^m(\theta^t)\right\}_{t\in[0,\infty)}$ be the optimal consumption path to problem (RP).

The following analysis closely follows Jeske (2006) with modifications to accommodate the limited commitment on domestic debt. Consider the types of residents with $\mu_n^m(\theta^t) > 0$. They fall further into two exclusive categories—group A with $v_{n_A}^m(\theta^r) = 0, \forall r$ and group B with $v_{n_B}^m(\theta^r) > 0$ for some r, where $r \in [t, \infty)$. Given their international participation constraints (5) hold with equality, residents at the brink of international debt default attain the same continuation value by staying with the Trade Equilibrium and returning to the Resident's International Autarky. Proposition 1 states the residents also consume the same amount of goods at every future history from θ^t.

Proposition 1 *In the Trade Equilibrium, if* $\mu_n^m(\theta^t) > 0$ *for some* $n, m,$ *and* θ^t*, then* $c_n^{m,D}(\theta^r)$ *and* $c_n^m(\theta^r)$ *are identical for all* θ^r *happening with positive probability after* θ^t*.*

Proof. *See Appendix A.2.* ■

This method achieves the same results above as found in Jeske's Proposition 1, which has been confirmed as extremely useful in proving all the following propositions. Proposition 2 states that, for any country or history, either every type is participation constrained in the international asset market or no type is constrained, even if the types have different endowment paths.

Proposition 2 *For all m and (θ^t, θ_{t+1}), either $q(\theta^t, \theta_{t+1}) > p^m(\theta^t, \theta_{t+1})$ and $\mu_n^m(\theta^t, \theta_{t+1}) > 0$, or $q(\theta^t, \theta_{t+1}) = p^m(\theta^t, \theta_{t+1})$ and $\mu_n^m(\theta^t, \theta_{t+1}) = 0$ for all n.*

Proof. *See Appendix A.3.* ∎

All types in country m must participate constrained in the international asset market at the same time. Otherwise, there is an arbitrage opportunity for non-constrained types to borrow internationally to their constraints and then relend at a higher interest rate to those constrained. This result is again the same as Jeske's Proposition 4. Removing his assumption of perfect enforcement on domestic debt does not alter the basic features of internationally constrained countries. However, within a constrained country, it will affect the tightness of international participation constraint (5) imposed on type n_B.

Comparing type n_A who are *never* domestic participation constrained in the post-default problem (RIA), n_B are *sometimes* domestic participation constrained at some future time, θ^r in (RIA). In other words, if residents of type n_B choose to default on international debt at θ^t and live in the Resident's International Autarky thereafter, they will again find themselves indifferent between reneging and repaying domestic debt at θ^r. In a constrained country, all types of group A share one equalized MRS, while different types of group B have different MRS. Proposition 3 says that n_B can neither borrow domestically from other types in group B with a higher MRS, nor lend domestically to all types in group A or to other types in group B with a lower MRS.

Proposition 3 *In the Trade Equilibrium, if $\nu_n^m(\theta^r) > 0$ with $r \in [t, \infty)$ in addition to $\mu_n^m(\theta^t) > 0$ for some n, m, and θ^t, then*

$$b_n^m(\theta^t) = \overline{B}_n^m(\theta^t),$$

where $\overline{B}_n^m(\theta^t)$ is determined by

$$\sum_{s=r}^{\infty} \beta^{s-r} \sum_{\theta^s|\theta^r} \pi(\theta^s|\theta^r) U(c_n^{m,D}(\theta^s, \theta^t, \overline{B}_n^m(\theta^t))) = A_n^m(\theta^r),$$

where $c_n^{m,D}(\theta^s, \theta^t, \overline{B}_n^m(\theta^t))$ is the optimal consumption at θ^s to the problem (RIA) with initial history θ^t, and initial bond holding, $\overline{B}_n^m(\theta^t)$.

Proof. *See Appendix A.4.* ∎

When it comes to decide the bond holding, $b_{n_B}^m(\theta^t)$ at θ^{t-1}, type n_B in a country that participates constrained in the international asset market's next period must buy a specified amount, $\overline{B}_{n_B}^m(\theta^r)$, even if it wants to borrow or *lend* more. This observation is critical to prove the next two propositions. Meanwhile, type n_A in the same country can freely choose its domestic bond holdings, but they do not want to deviate as long as an optimum has been arrived.

In general, the international participation constraint (5) makes the problem (RP) non-convex. Proposition 4 justifies the sufficiency of the first-order-condition approach for a global maximum. The main idea is similar to Jeske. First, define an alternative maximization problem with the same objective function and a convex constraint set that is a superset of the non-convex constraint set in the original non-convex problem. Then, show a solution to the original problem is also affordable and individually rational in the alternative convex problem. It turns out that both problems have identical first order conditions; thus, the same optimal solutions.

Proposition 4 *For all n, m, together with a transversality condition*

$$\lim_{T\to\infty} \beta^T \sum_{\theta^T} U'(c_n^m(\theta^T))\pi(\theta^T) \left[b_n^m(\theta^T) + f_n^m(\theta^T)\right] = 0,$$

first order conditions (7-9) are sufficient to characterize the maximum of the resident's problem (RP).

Proof. *See Appendix A.5.* ■

Proposition 5 shows how domestic and international bond prices are determined in equilibrium.

Proposition 5 *In Trade Equilibrium, for all n, m, and $\left(\theta^t, \theta_{t+1}\right)$ with $t \in [0, \infty)$,*

(I) the international bond price,

$$q(\theta^t, \theta_{t+1}) = \max_{m=1,\dots,M;n=1,\dots,N} \left\{ \beta \frac{U'(c_n^m(\theta^t, \theta_{t+1}))}{U'(c_n^m(\theta^t))} \pi(\theta_{t+1}|\theta^t) \right\},$$

(II) the domestic bond price in country $m \in \{1, 2, ..., M\}$,

$$p^m(\theta^t, \theta_{t+1}) = \min_{n=1,\dots,N} \left\{ \beta \frac{U'(c_n^m(\theta^t, \theta_{t+1}))}{U'(c_n^m(\theta^t))} \pi(\theta_{t+1}|\theta^t) \right\},$$

and finally

(III) the relationship between the international bond price and all M domestic bond prices is

$$q(\theta^t, \theta_{t+1}) = \max_{m=1,\dots,M} \left\{ p^m(\theta^t, \theta_{t+1}) \right\}.$$

Proof. *See Appendix A.6.* ■

In Proposition 5(I), the international bond price is equal to the highest MRS among all N types and across all M countries so that repaying would not hurt debtors as much as living isolated from the world. Part (II) says the domestic bond price in m is equal to the lowest MRS among all N types in m. This rule overturns the results from closed economy models, in which price must be the highest to guarantee an incentive to fulfill the debtor's obligations. The reason is the domestic interest rate, as a device to ensure repayment, is no longer needed in this paper. On the other hand, the domestic interest rate plays another role of punishing international debt defaulters harsher; hence, raising the borrowing ceiling. Take any constrained country for example. When its residents are contemplating international debt default, they find themselves more miserable in Resident's International Autarky, since the domestic interest rate is higher than the level they would have expected. Conversely, the country with $\mu_n^m(\theta^t, \theta_{t+1}) = 0, \forall n$ is a special case, since the MRS are equalized among all N types. Moreover, its domestic bond price must be equal to the international bond price to rule out arbitrage possibilities. Combining the above rules reveals the relationship in part (III).

3 Welfare Analysis

In this section, Jeske's setup is presented in the notation used by this paper as a benchmark. Next, the weighted average aggregate welfare in any country is shown to strictly improve, when the world is switched to the setup discussed in this paper. Then, another alternative method to elevate weighted average aggregate welfare by centralization is introduced. Finally, the conditions on the endowment structure, under which this new model does a better job, are identified.

3.1 A Comparison with the Benchmark

Recall Jeske's studies (2001, 2006) with perfect enforcement on domestic debt. After type n residents in country m renege on international debt at θ^t, their value can be represented as

$$V_n^{m,J}(\theta^t, b_n^m(\theta^t)) \equiv \max_{\{c_n^m(\theta^r), b_n^m(\theta^r, \theta_{r+1})\}_{r\in[t,\infty)}} \sum_{r=t}^{\infty} \beta^{r-t} \sum_{\theta^r|\theta^t} \pi(\theta^r|\theta^t) U(c_n^m(\theta^r)), \tag{RIA$^{\text{J}}$}$$

subject to the budget constraint,

$$e_n^m(\theta^r) + b_n^m(\theta^r) \geqslant c_n^m(\theta^r) + \sum_{\theta_{r+1}} p^m(\theta^r, \theta_{r+1}) b_n^m(\theta^r, \theta_{r+1}),$$

and the no-Ponzi condition,

$$b_n^m(\theta^r, \theta_{r+1}) \geqslant -\overline{B},$$

with $b_n^m(\theta^t)$ and $\{p^m(\theta^r, \theta_{r+1})\}_{r\in[t,\infty)}$ given, for all histories θ^r and all states (θ^r, θ_{r+1}) with $r \in [t, \infty)$.

Again, at date 0 before any default can happen, Jeske's resident's problem is

$$\max_{\{c_n^m(\theta^t), f^m(\theta^t, \theta_{t+1})\}_{t\in[0,\infty)}} \sum_{t=0}^{\infty} \beta^t \sum_{\theta^t} \pi(\theta^t) U(c_n^m(\theta^t)), \tag{RP$^{\mathrm{J}}$}$$

subject to the budget constraint,

$$\begin{aligned} & e_n^m(\theta^t) + b_n^m(\theta^t) + f_n^m(\theta^t) \\ \geqslant\ & c_n^m(\theta^t) + \sum_{\theta_{t+1}} p^m(\theta^t, \theta_{t+1}) b_n^m(\theta^t, \theta_{t+1}) + \sum_{\theta_{t+1}} q(\theta^t, \theta_{t+1}) f_n^m(\theta^t, \theta_{t+1}), \end{aligned}$$

the participation constraint in the international asset market,

$$\sum_{r=t}^{\infty} \beta^{r-t} \sum_{\theta^r|\theta^t} \pi(\theta^r|\theta^t) U(c_n^m(\theta^r)) \geqslant V_n^{m,J}(\theta^t, b_n^m(\theta^t)), \tag{10}$$

and the no-Ponzi conditions,

$$b_n^m(\theta^t, \theta_{t+1}) \geqslant -\overline{B}, f_n^m(\theta^t, \theta_{t+1}) \geqslant -\overline{F},$$

with initial bond holdings,

$$b_n^m(\theta^0) \text{ and } f_n^m(\theta^0),$$

and the price sequence,

$$\left\{p^m(\theta^t, \theta_{t+1}), q(\theta^t, \theta_{t+1})\right\}_{t\in[0,\infty)} \text{ given,}$$

for all histories θ^t and all states $\left(\theta^t, \theta_{t+1}\right)$ with $t \in [0, \infty)$.

Problem (RP$^{\mathrm{J}}$)'s corresponding competitive equilibrium can be defined similarly to Definition 1. For any m, all types in the new model achieve a higher or at least the same welfare level than the benchmark's setup. A useful way to think about the result is that type n_B is now confronted with more severe penalties after an international debt default (lower continuation value in Resident's International Autarky). Assume a small open economy, m, stops enforcing domestic debt after a bad shock. Type n_A's optimization problem can still be defined by (RP$^{\mathrm{J}}$); thus, their welfare level stays the same. However, type n_B's optimization problem shifts from (RP$^{\mathrm{J}}$) to (RP). Their original optimal allocations in (RP$^{\mathrm{J}}$) are both affordable in (RP), since bond prices determined by type n_A remain unchanged, and individual rational in (RP), since international participation constraints (5) are less restricted and the newly added domestic participation constraints (6) are superfluous. If there do exist type n_B residents whose domestic participation constraints (2) bind after some future histories in (RIA) when the host country is internationally constrained after a present history in (RP), and if there is positive international capital inflow at this present history in (RP), then type n_B

residents can do strictly better after the shock by relaxing their international participation constraints (5) at the present history in (RP). Summing up the welfare levels of all types with exogenously provided weights assigned to each type, the country's aggregate welfare improves. The following Proposition 6 formalizes this argument.

Proposition 6 *Assume the welfare-weighted index is given by* $\{\varphi_n^m\}_{n=1}^N$ *with* $\varphi_n^m \in \mathbb{R}_{++}$ *for all types in country* m*. Let* $\left\{c_n^m\left(\theta^t\right), b_n^m(\theta^t,\theta_{t+1}), f_n^m(\theta^t,\theta_{t+1})\right\}_{t\in[0,\infty)}$ *solve the resident's problem (RP) in Section 2, and* $\left\{c_n^{m,J}\left(\theta^t\right), b_n^{m,J}(\theta^t,\theta_{t+1}), f_n^{m,J}(\theta^t,\theta_{t+1})\right\}_{t\in[0,\infty)}$ *solve the resident's problem (RP*^{J}*) in Jeske (2006). Then,*

$$\sum_{n=1}^N \varphi_n^m \sum_{t=0}^\infty \beta^t \sum_{\theta^t} \pi(\theta^t) U(c_n^m(\theta^t)) \geqslant \sum_{n=1}^N \varphi_n^m \sum_{t=0}^\infty \beta^t \sum_{\theta^t} \pi(\theta^t) U(c_n^{m,J}(\theta^t)),$$

with strict inequality if there is a history (θ^t,θ_{t+1})*, at which* $\frac{U'\left(c_n^m(\theta^t,\theta_{t+1})\right)}{U'(c_n^m(\theta^t))} \neq \frac{U'\left(c_1^m(\theta^t,\theta_{t+1})\right)}{U'\left(c_1^m(\theta^t)\right)}$ *for some* n *in* m *(domestic imperfect sharing).*

Proof. *See Appendix A.7.* ■

3.2 A Comparison of Improvement

Suppose country m is taken over by a benevolent planner, who trades internationally and allocate consumption domestically; whereas, residents do not access any markets. Therefore, a limited commitment problem within borders is eliminated, but the planner can still default on national debt across borders if autarky utility turns out better. Assume the planner uses the same welfare weights as before; hence, the national autarky value

$$V^m(\theta^t) \equiv \max_{\{c_n^m(\theta^r)\}_{r\in[t,\infty)}} \sum_{n=1}^N \varphi_n^m \sum_{r=t}^\infty \beta^{r-t} \sum_{\theta^r|\theta^t} \pi(\theta^r|\theta^t) U(c_n^m(\theta^r)), \tag{PA}$$

subject to the resource constraint,

$$\sum_{n=1}^N e_n^m(\theta^r) \geqslant \sum_{n=1}^N c_n^m(\theta^r),$$

for all histories θ^r and all states (θ^r,θ_{r+1}) with $r\in[t,\infty)$.

Before a default, the planner's problem is

$$\max_{\{c_n^m(\theta^t)\}_{t\in[0,\infty)}} \sum_{n=1}^N \varphi_n^m \sum_{t=0}^\infty \beta^t \sum_{\theta^t} \pi(\theta^t) U(c_n^m(\theta^t)), \tag{PP}$$

subject to the budget constraint,

$$\sum_{n=1}^N e_n^m(\theta^t) + f^m(\theta^t) \geqslant \sum_{n=1}^N c_n^m(\theta^t) + \sum_{\theta_{t+1}} q(\theta^t,\theta_{t+1}) f^m(\theta^t,\theta_{t+1}),$$

country's participation constraint in the international asset market,

$$\sum_{n=1}^{N}\varphi_n^m\sum_{r=t}^{\infty}\beta^{r-t}\sum_{\theta^r|\theta^t}\pi(\theta^r|\theta^t)U(c_n^m(\theta^r))\geqslant V^m(\theta^t), \tag{11}$$

and the no-Ponzi condition,

$$f^m(\theta^t,\theta_{t+1})\geqslant -\overline{F},$$

with $f^m(\theta^0)$ and $\left\{q(\theta^t,\theta_{t+1})\right\}_{t\in[0,\infty)}$ given, for all histories θ^t and all states $\left(\theta^t,\theta_{t+1}\right)$ with $t\in[0,\infty)$.

Its corresponding competitive equilibrium can be defined similarly to Definition 1 only without a domestic asset market. In conclusion, to strictly improve the benchmark aggregate welfare, one must determine a method to decrease the weighted average of post-default values (RIA) across all types. Jeske suggested this can be accomplished by prohibiting private international borrowing; hence, internalizing the associated externalities on domestic prices, as long as the benchmark situation is not autarky for all histories.[6] However, the same goal could also be achieved with less radical methods to retain the more realistic private debt framework. Wright (2006) used a system of international borrowing subsidies to mimic the constrained optimum under centralization. This paper uses a limited commitment on domestic risk sharing. Next, Proposition 7 says my model can outperform centralized arrangements if the losses from domestic friction outweigh the gains from intermediary borrowing in the maximization problem after an international debt default.

Proposition 7 *Let* $\left\{c_n^m\left(\theta^t\right), b_n^m(\theta^t,\theta_{t+1}), f_n^m(\theta^t,\theta_{t+1})\right\}_{t\in[0,\infty)}$ *solve the resident's problem (RP) in Section 2, and* $\left\{c_n^{m,P}\left(\theta^t\right), f^m(\theta^t,\theta_{t+1})\right\}_{t\in[0,\infty)}$ *solve the planner's problem (PP). Then,*

$$\sum_{n=1}^{N}\varphi_n^m\sum_{t=0}^{\infty}\beta^t\sum_{\theta^t}\pi(\theta^t)U(c_n^m(\theta^t))>\sum_{n=1}^{N}\varphi_n^m\sum_{t=0}^{\infty}\beta^t\sum_{\theta^t}\pi(\theta^t)U(c_n^{m,P}(\theta^t)),$$

if there is a history (θ^t,θ_{t+1}) *such that*

(I) future endowment path $\{e_n^m(\theta^r)\}_{r\in[t,\infty)}, \forall n$ *in country* m *supports the following inequality*

$$\sum_{n=1}^{N}\varphi_n^m V_n^m(\theta^t, b_n^m(\theta^t))<V^m(\theta^t),$$

and

(II) domestic imperfect sharing, $\frac{U'\left(c_n^m(\theta^t,\theta_{t+1})\right)}{U'(c_n^m(\theta^t))}\neq\frac{U'\left(c_1^m(\theta^t,\theta_{t+1})\right)}{U'\left(c_1^m(\theta^t)\right)}$, *for some* n *in* m.

Proof. *See Appendix A.8.* ■

[6] Essentially, this non-autarkic condition is equivalent to the domestic imperfect sharing condition in Proposition 6 except the latter one implicitly incorporates an additional assumption of type n_B's existence.

4 Numerical Example

The main results are illustrated through an example in this section. Consider a world with country 1 and 2, each is populated by a unit mass of residents with static preference $U(c) = \log(c)$, and discount factor $\beta \in (0, 1)$. Residents born at period 0 live forever and time t is discrete. One sort of non-storable-consumption goods from endowment is traded every period, according to risk sharing contracts signed before birth. The country's initial endowment could be either high with $1 + y$ or low with $1 - y$, and the beginning endowment in country 1 differs from 2. After this initial structure is revealed, aggregate endowment then alternates between high and low deterministically in both countries. Residents are labeled as either type A, who face an idiosyncratic shock with positive ε in high state, $1 + y + \varepsilon$, and negative ε in low state, $1 - y - \varepsilon$, or type B, who face the opposite shock, $1 + y - \varepsilon$ and $1 - y - \varepsilon$. It is assumed that $y, \varepsilon \in (0, 1)$ and $y \geqslant \varepsilon$, capturing the idea that income fluctuation across countries is more volatile than income fluctuation within a country.[7] The key difference between types is that B has a smoother endowment path.

Prior to period 0, the timeline of contracting is as follows. First, residents within the same country enter into a domestic risk-sharing contract. Second, a coin flip determines the type of half random residents in both countries. Then, domestic debt obligations will be fufilled under the assumption of domestic enforcement or subject to default with a pervasive commitment problem. Third, domestic and foreign agents agree on an international risk-sharing contract. Eventually, another independent coin flip determines countries' initial endowments. After this, agents will not deviate from international agreement as long as they are better off than autarky.

Without loss of generality, suppose country 1 starts with a high state at $t = 0$. Table 1 summarizes the endowment structure at even numbered periods, while Table 2 includes all odd numbered periods. Country 1 is participation constrained in te international asset market in the even numbered period, including type B with a relatively smaller endowment. As in the formal model, country is denoted by superscript and type by subscript. Let z denote the consumption deviation for a representative resident of either type in country 1, her lifetime preference can be written as

$$u^1(z) = \sum_{t=0}^{\infty} \beta^t \log(1 + (-1)^t z) = \frac{\beta}{1 - \beta} \left[\log(1 + z) + \beta \log(1 - z)\right].$$

[7] $y < \varepsilon$ makes the limited commitment problem on domestic debt irrelevant since domestic risk sharing is always perfect in this situation, even without legal enforcement.

Rescale it to obtain

$$u^1(z) = \log(1+z) + \beta \log(1-z).$$

Similarly, residents in country 2 have utility,

$$u^2(z) = \log(1-z) + \beta \log(1+z).$$

Note the above $u^m(\cdot), m \in \{1, 2\}$ represents the ex-post utility given the country's initial endowment. Ex ante when aggregate endowment structure is not yet revealed, any one representative agent in either country has an expected lifetime preference,

$$E\left[u(z)\right] = \frac{1}{2}u^1(z) + \frac{1}{2}u^2(z).$$

Welfare comparison is thus based on this ex-ante level. Since $E\left[u(z)\right]$ is strictly decreasing in z, a smaller consumption deviation, z, means more international risk sharing; hence, higher welfare.

4.1 Private Borrowing with Domestic Enforcement

Because domestic debts are perfectly enforced, different types in the same country can be treated as a single type since they consume the same amount of goods every period. After the subscript is removed, the consumption pattern $c^m(t)$ for residents in country m is

$$\begin{aligned} c^1(2k) &= 1 + x^J, c^1(2k+1) = 1 - x^J; \\ c^2(2k) &= 1 - x^J, c^2(2k+1) = 1 + x^J, \end{aligned}$$

where $k \in \mathbb{N}$, the set of non-negative integers. By Lemma 2, the present value of all future payments from the constrained country to the unconstrained country is zero, when discounted by Arrow-Debreu domestic bond prices.

$$\frac{x^J - y + q\left(y - x^J\right)}{1 - pq} = 0, \tag{12}$$

where q (p) denotes the domestic bond price in the county unconstrained (constrained) in international asset market the next period.[8]

In other words, q can be found for residents who consume $1 + x^J$ today and $1 - x^J$ tomorrow,

$$q = \beta \frac{1 + x^J}{1 - x^J}, \tag{13}$$

[8] See the derivation in Appendix A.9.

and residents who consume just the opposite face domestic price,

$$p = \beta \frac{1 - x^J}{1 + x^J}.$$

Therefore, the international bond price $q(t) = q, \forall t$, and the domestic bond price $p^m(t)$ jumps between p and q over time.

$$\begin{aligned} p^1(t) &= \begin{cases} p \text{ for } t = 2k; \\ q \text{ for } t = 2k+1. \end{cases} \\ p^2(t) &= \begin{cases} q \text{ for } t = 2k; \\ p \text{ for } t = 2k+1. \end{cases} \end{aligned} \tag{14}$$

There are two solutions to Eq. (12). The first is autarky, or $x^J = y$, while the second requires $q = 1$, which further implies $x^J = \frac{1-\beta}{1+\beta}$ using Eq. (13). Let x^J be the benchmark risk sharing level. Later, two alternative setups that can both improve welfare—centralization in Section 4.2 and no enforcement on domestic debt in Section 4.3—will be shown. Section 4.4 discusses the best solution of these two setups.

4.2 Centralized Borrowing

In a centralized economy, the government prohibits private debt. Instead, it borrows in an international asset market on behalf of its residents, and then apportions the total resources among them equally. One can aggregate each country into a representative agent whose consumption $c^m(t)$ has the following pattern,

$$\begin{aligned} c^1(2k) &= 1 + x^c, c^1(2k+1) = 1 - x^c; \\ c^2(2k) &= 1 - x^c, c^2(2k+1) = 1 + x^c, \end{aligned}$$

where x^c is the smallest deviation satisfying country 1's international participation constraint in an even period.

$$x^c = \min_{z \geqslant 0} \{z : \log(1+z) + \beta \log(1-z) \geq \log(1+y) + \beta \log(1-y)\}.$$

To support some international risk sharing across borders, aggregate endowment, y, must satisfy two restrictions, given the value of an exogenous discount factor, β. The first restriction is

$$-\frac{\log(1+y)}{\log(1-y)} > \beta, \tag{15}$$

otherwise consumption is fully smoothed, $x^c = 0$. The second restriction is

$$y > \frac{1-\beta}{1+\beta}, \tag{16}$$

otherwise autarky is the highest utility one can achieve and there is no trade in equilibrium, $x^c = y$.

If y does lie within the range defined by Eqs. (15) and (16), then we observe $x^c < \frac{1-\beta}{1+\beta}$ from Figure 1. Since $x^J = \frac{1-\beta}{1+\beta}$ or y, centralized borrowing is welfare superior than Jeske's decentralized setup.

$$E\left[u(x^c)\right] > E\left[u(x^J)\right].$$

4.3 Private Borrowing with Pervasive Risk of Default

Next, remove the assumption of perfect enforcement on domestic debt. Optimal consumption now alternates between not only different types within a country, but also across countries. By symmetry, $c_n^m(t)$ takes the following values,

$$\begin{aligned}
c_A^1(2k) &= 1 + x + \varepsilon^p, c_A^1(2k+1) = 1 - x - \varepsilon^p; \\
c_B^1(2k) &= 1 + x - \varepsilon^p, c_B^1(2k+1) = 1 - x + \varepsilon^p; \\
c_A^2(2k) &= 1 - x - \varepsilon^p, c_A^2(2k+1) = 1 + x + \varepsilon^p; \\
c_B^1(2k) &= 1 - x + \varepsilon^p, c_B^1(2k+1) = 1 + x - \varepsilon^p,
\end{aligned}$$

where $\varepsilon^p > 0$ indicating imperfect domestic risk sharing.[9]

Type B with a less volatile deviation is also domestically participation constrained when the host country is as a whole internationally participation constrained. x and ε^p are jointly determined by international participation constraints for both types,

$$\begin{cases} \frac{(x+\varepsilon^p)-y+q[y-(x+\varepsilon^p)]}{1-pq} = 0 \text{ for type A;} \\ \frac{(x-\varepsilon^p)-y+q[y-(x-\varepsilon^p)]}{1-pq} = 0 \text{ for type B,} \end{cases} \tag{17}$$

and an additional constraint for type B that ensures no incentive to default on the domestic debt.

$$x - \varepsilon^p = \min_{z \geqslant 0} \left\{ z : \begin{array}{c} \log(1+z) + \beta \log(1-z) \\ \geqslant \log(1+y-\varepsilon) + \beta \log(1-y+\varepsilon) \end{array} \right\}. \tag{18}$$

The international bond price always equals the highest MRS across the world,

$$q = \beta \frac{1 + x + \varepsilon^p}{1 - x - \varepsilon^p}, \tag{19}$$

[9] $\varepsilon^p = 0$ returns to the same risk sharing pattern as in Jeske's setup. To make the domestic commitment problem interesting, assume ε^p to be strictly positive throughout this section.

and the domestic bond price in a constrained country equals to the lowest MRS within that country,

$$p = \beta \frac{1 - x - \varepsilon^p}{1 + x + \varepsilon^p}.$$

Domestic bond price sequences in country 1 and 2 will look the same as Eq. (14) in Section 4.1.[10]

Solving Eq. (17) gives one unique solution, $q = 1$, which further implies $x + \varepsilon^p = \frac{1-\beta}{1+\beta}$ for type A using Eq. (19). As seen in Figure 1 where $y - \varepsilon > \frac{1-\beta}{1+\beta}$, domestic participation constraint (18) requires $x - \varepsilon^p < \frac{1-\beta}{1+\beta}$ for type B.[11] No enforcement on domestic debt improves the benchmark welfare level as well.

$$\frac{1}{2} E\left[u(x + \varepsilon^p)\right] + \frac{1}{2} E\left[u(x - \varepsilon^p)\right] > E\left[u(x^J)\right].$$

4.4 A Comparison of Welfare Improvement

Since both centralization and this study's setup can strictly improve welfare in Jeske's benchmark model, the question then becomes which one improves the benchmark model better. The answer depends upon the distance between aggregate endowment parameter, y, and idiosyncratic endowment parameter, ε. When y is relatively larger than ε, centralization results in a greater increment. On the other hand, if ε lies within a close neighborhood of y, a pervasive commitment problem does a better job. To understand this, first define the boundary of the neighborhood $\psi < \frac{1-\beta}{1+\beta}$ implicitly by

$$\frac{1}{2} E\left[u(\frac{1-\beta}{1+\beta})\right] + \frac{1}{2} E\left[u(\psi)\right] = E\left[u(x^c)\right].$$

Then, recall the implications of domestic participation constraint (18).

$$\begin{aligned} y - \varepsilon &\in \left(\frac{1-\beta}{1+\beta}, y\right) \Rightarrow x^c < x - \varepsilon^p < \frac{1-\beta}{1+\beta}; \\ y - \varepsilon &\in \left(0, \frac{1-\beta}{1+\beta}\right) \Rightarrow x - \varepsilon^p = y - \varepsilon < \frac{1-\beta}{1+\beta}. \end{aligned}$$

Now, compare the aggregate utility $\frac{1}{2} E\left[u(x + \varepsilon^p)\right] + \frac{1}{2} E\left[u(x - \varepsilon^p)\right]$ and centralized utility $E\left[u(x^c)\right]$. If $y - \varepsilon \in \left(\frac{1-\beta}{1+\beta}, y\right)$, then $E\left[u(x^c)\right]$ is greater, since $x + \varepsilon^p, x - \varepsilon^p > x^c$, and $E[u(\cdot)]$ is strictly decreasing in its

[10] In contrast to the formal model, the MRS is not equalized within a unconstrained country in this example. Specifically, type A has a larger MRS than Type B in country 1 at period 0

$$\beta \frac{1 + x + \varepsilon^p}{1 - x - \varepsilon^p} > \beta \frac{1 + x - \varepsilon^p}{1 - x + \varepsilon^p}.$$

The reason this occurs is consumption jumps back and forth every two periods.

[11] If $y - \varepsilon = \frac{1-\beta}{1+\beta}$, then Eq. (18) tstates $x - \varepsilon^p = \frac{1-\beta}{1+\beta}$, which implies $\varepsilon^p = 0$ together with $x + \varepsilon^p = \frac{1-\beta}{1+\beta}$. This situation is ruled out by the assumption of $\varepsilon^p > 0$. If $y - \varepsilon < \frac{1-\beta}{1+\beta}$, then $x - \varepsilon^p = y - \varepsilon$, which is again by Eq. (18).

argument. If $y-\varepsilon \in \left[\psi, \frac{1-\beta}{1+\beta}\right)$, then replacing $x-\varepsilon^p$ with $y-\varepsilon$ implies that $E\left[u(x^c)\right]$ is still greater or equal to the aggregate utility level. However, if $y-\varepsilon \in [0,\psi)$, then $\frac{1}{2}E\left[u(x+\varepsilon^p)\right]+\frac{1}{2}E\left[u(x-\varepsilon^p)\right]$ is greater by the definition of threshold value, ψ. In summary, the conclusion is given that $\varepsilon^p > 0$ and $y \geqslant \varepsilon$,

$$\frac{1}{2}E\left[u(x+\varepsilon^p)\right]+\frac{1}{2}E\left[u(x-\varepsilon^p)\right]\begin{cases} \leqslant E\left[u(x^c)\right] & \text{if } y-\varepsilon \geqslant \psi \\ > E\left[u(x^c)\right] & \text{if } y-\varepsilon < \psi \end{cases}.$$

This welfare ordering is generally true. Nevertheless, this might change when either Eq. (15) or Eq. (16) is relaxed. Consider the extreme case when the country's aggregate income fluctuation is extremely volatile, i.e., $-\frac{\log(1+y)}{\log(1-y)} < \beta$. Complete international risk sharing, which can never be an equilibrium result in a private debt environment, is achieved in the centralized arrangement with highest welfare. On the other hand, when $y < \frac{1-\beta}{1+\beta}$, then both centralization and Jeske's model lead to the same equilibrium of autarky with complete domestic risk sharing $x^c = x^J = y$. This study's model will cause autarky with no domestic risk sharing, $x = y$ and $\varepsilon^p = \varepsilon$, which is the worst scenario one can expect.

5 Conclusion

This paper developed an open economy model with heterogeneous agents sharing endowment volatility both within and across border. This model is built on Jeske's private borrowing setup, except for relaxing his key assumption of perfect enforcement on domestic debt. Therefore, risk of repudiation is pervasive in all debt contracts. International debt differs from domestic debt in not only price, but also default punishments. Specifically, international debt defaulters are excluded only from the international asset market, while domestic debt defaulters are denied from all asset markets.

The main contribution of this paper is to show an economy with a pervasive enforcement problem does better at gaining trust in an international capital market than an economy with enforcement problems on foreign debt alone. The reason is that the penalty on international debt default is at least as harsh as in Jeske's model and harsher for other types with small endowment volatility. Thus, more capital inflow and higher welfare can be supported for these lucky types. Negative externality originated from individual decisions is mitigated in some sense. In contrast to fully internalizing by a radical method, such as capital control, this paper keeps the private debt framework, improves the aggregate welfare, and outperforms capital control in certain countries. Another result is about asset pricing. In this model, the domestic interest rate equals the reciprocal of the lowest MRS in countries that participate constrained internationally. That is, some agents living in these countries are lending constrained domestically. This overthrows the well-established argument that interest rates should be the lowest to induce repayment in an environment without legal enforcement. This contradictory result is due to the crucial ingredient of this paper—equilibrium domestic debt default can never happen before an international debt default. Its repayment is secured, as long as the mechanism prevents attempted international debt default. For countries whose international participation constraint is superfluous, their domestic interest rate equals the prevailing international interest rate as cited in previous literature.

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A Appendix

A.1 Trade Equilibrium Solution

The Lagrangian of the resident's problem (RP) is (drop the superscript and subscript for simplicity)

$$\begin{aligned} L_{RP} &= \sum_{t=0}^{\infty}\beta^t \sum_{\theta^t}\pi(\theta^t)U(c(\theta^t)) + \\ &+\sum_{t=0}^{\infty}\sum_{\theta^t}\kappa(\theta^t)\left[e(\theta^t)+b(\theta^t)+f(\theta^t)-c(\theta^t)\right] \\ &-\sum_{t=0}^{\infty}\sum_{\theta^t}\kappa(\theta^t)\left[p(\theta^t,\theta_{t+1})b(\theta^t,\theta_{t+1})+q(\theta^t,\theta_{t+1})f(\theta^t,\theta_{t+1})\right] \\ &+\sum_{t=0}^{\infty}\sum_{\theta^t}\mu(\theta^t)\left[\sum_{s=t}^{\infty}\beta^{s-t}\sum_{\theta^s|\theta^t}\pi(\theta^s|\theta^t)U(c(\theta^s))-V(\theta^t,b_n^m(\theta^t))\right]. \end{aligned}$$

First order conditions with respect to $c(\theta^t), b(\theta^t,\theta_{t+1})$ and $f(\theta^t,\theta_{t+1})$ are, respectively, Eqs. (7-9). Rearrange Eqs. (7) to obtain

$$\kappa(\theta^t) = \beta^{r-t}\pi(\theta^t)U'(c(\theta^t))\left[1+\sum_{s=0}^{t}\sum_{\theta^t|\theta^s}\mu(\theta^s)\beta^{-s}\frac{\pi(\theta^t|\theta^s)}{\pi(\theta^t)}\right]. \quad \text{(A.1)}$$

Before using Eq. (8), the post-default optimization problem (RIA) was solved to achieve a closed form of its envelope condition, $\frac{dV(\theta^t,\theta_{t+1},b(\theta^t,\theta_{t+1}))}{db(\theta^t,\theta_{t+1})}$.

To solve (RIA) with an initial history θ^t, first write the Lagrangian.

$$\begin{aligned} L_{RIA} &= \sum_{r=t}^{\infty}\beta^{r-t}\sum_{\theta^r|\theta^t}\pi(\theta^r|\theta^t)U(c(\theta^r)) \\ &+\sum_{r=t}^{\infty}\sum_{\theta^r|\theta^t}\lambda(\theta^r)\left[e(\theta^r)+b(\theta^r)-c(\theta^r)-\sum_{\theta_{r+1}}p(\theta^r,\theta_{r+1})b(\theta^r,\theta_{r+1})\right] \\ &+\sum_{r=t}^{\infty}\sum_{\theta^r|\theta^t}\nu(\theta^r)\left[\sum_{s=r}^{\infty}\beta^{s-r}\sum_{\theta^s|\theta^r}\pi(\theta^s|\theta^r)U(c(\theta^s))-A_n^m(\theta^r)\right]. \end{aligned}$$

Rewrite the first order condition with respect to $c(\theta^r)$ to obtain

$$\lambda(\theta^r) = \beta^{r-t}\pi(\theta^r|\theta^t)U'(c(\theta^r))\left[1+\sum_{s=t}^{r}\sum_{\theta^r|\theta^s}\nu(\theta^s)\beta^{t-s}\frac{\pi(\theta^r|\theta^s)}{\pi(\theta^r|\theta^t)}\right]. \quad \text{(A.2)}$$

Given initial domestic bond holdings, $b_n^m(\theta^t)$, the Envelope Theorem yields

$$\frac{dV(\theta^t,b(\theta^t))}{db(\theta^t)} = \frac{\partial L_{RIA}}{\partial b(\theta^t)} = \lambda(\theta^t). \quad \text{(A.3)}$$

Combine Eqs. (A.2) and (A.3).

$$\begin{aligned}\frac{dV(\theta^t, b(\theta^t))}{db(\theta^t)} &= \beta^{t-t}\pi(\theta^t|\theta^t)U'(c^D(\theta^t))\left[1+\sum_{s=t}^{t}\sum_{\theta^t|\theta^s}\nu(\theta^s)\beta^{t-s}\frac{\pi(\theta^t|\theta^s)}{\pi(\theta^t|\theta^t)}\right] \\ &= \left[1+\nu(\theta^t)\right]U'(c^D(\theta^t)).\end{aligned}$$

Iterate $\frac{dV(\theta^t,b(\theta^t))}{db(\theta^t)}$ one period forward to generate

$$\frac{dV(\left(\theta^t,\theta_{t+1}\right), b(\theta^t,\theta_{t+1}))}{db(\theta^t,\theta_{t+1})} = \left[1+\nu(\theta^t,\theta_{t+1})\right]U'(c^D(\theta^t,\theta_{t+1})). \tag{A.4}$$

Now, continue with the problem (RP). Substitute Eq. (A.4) into Eq. (8) and solve for the domestic bond price together with Eq. (A.1).

$$\begin{aligned}p(\theta^t,\theta_{t+1}) &= \frac{\kappa(\theta^t,\theta_{t+1})-\mu(\theta^t,\theta_{t+1})\frac{dV(\left(\theta^t,\theta_{t+1}\right),b(\theta^t,\theta_{t+1}))}{db(\theta^t,\theta_{t+1})}}{\kappa(\theta^t)} \\ &= \frac{\kappa(\theta^t,\theta_{t+1})-\mu(\theta^t,\theta_{t+1})\left[1+\nu(\theta^t,\theta_{t+1})\right]U'(c^D(\theta^t,\theta_{t+1}))}{\kappa(\theta^t)} \\ &= \beta\frac{U'(c(\theta^t,\theta_{t+1}))}{U'(c(\theta^t))}\pi(\theta_{t+1}|\theta^t)\frac{1+A_2-\left(1+\nu(\theta^t,\theta_{t+1})\right)A_1}{1+A_3},\end{aligned} \tag{A.5}$$

where

$$\begin{aligned}A_1 &= \mu(\theta^t,\theta_{t+1})\beta^{-t-1}\frac{U'(c^D(\theta^t,\theta_{t+1}))}{U'(c(\theta^t,\theta_{t+1}))}\frac{1}{\pi(\theta^t,\theta_{t+1})}; \\ A_2 &= \sum_{s=0}^{t+1}\sum_{\theta^t,\theta_{t+1}|\theta^s}\mu(\theta^s)\beta^{-s}\frac{\pi(\theta^t,\theta_{t+1}|\theta^s)}{\pi(\theta^t,\theta_{t+1})}; \\ A_3 &= \sum_{s=0}^{t}\sum_{\theta^t|\theta^s}\mu(\theta^s)\beta^{-s}\frac{\pi(\theta^t|\theta^s)}{\pi(\theta^t)}.\end{aligned}$$

Finally, solve for the international bond price with Eqs. (9) and (A.1).

$$\begin{aligned}q(\theta^t,\theta_{t+1}) &= \frac{\kappa(\theta^t,\theta_{t+1})}{\kappa(\theta^t)} \\ &= \beta\frac{U'(c(\theta^t,\theta_{t+1}))}{U'(c(\theta^t))}\pi(\theta_{t+1}|\theta^t)\frac{1+A_2}{1+A_3}.\end{aligned} \tag{A.6}$$

A.2 Proof of Proposition 1

Imagine an Arrow-Debreu setup, where a domestic asset market exists at period 0 for all kinds of bonds that mature at any future period. Denote $P^m(\theta^r) = P^m(\theta^{r-1})p^m(\theta^{r-1},\theta_r) = \prod_{s=0}^{r} p^m(\theta^s)$, the forward price for a r-period matured domestic contingent bond at period 0, where $r \in (0,\infty)$ and $P^m(\theta^0) = p^m(\theta^0) = 1$. The proof proceeds in three steps:

STEP 1: Redefine the Resident's International Autarky problem (RIA) started at θ^t as follows

$$V_n^m(\theta^t, b_n^m(\theta^t)) \equiv \max_{\{c_n^m(\theta^r)\}_{r\in[t,\infty)}} \sum_{r=t}^{\infty} \beta^{r-t} \sum_{\theta^r|\theta^t} \pi(\theta^r|\theta^t) U(c_n^m(\theta^r)), \tag{RIA$^{\mathrm{F}}$}$$

subject to the summation of all future budget constraints after history θ^t being discounted to period 0,

$$\sum_{r=t}^{\infty} \sum_{\theta^r|\theta^t} P^m(\theta^r) c_n^m(\theta^r) = \sum_{r=t}^{\infty} \sum_{\theta^r|\theta^t} P^m(\theta^r) e_n^m(\theta^r) + P^m(\theta^t) b_n^m(\theta^t),$$

and the participation constraint in the domestic asset market,

$$\sum_{s=r}^{\infty} \beta^{s-r} \sum_{\theta^s|\theta^r} \pi(\theta^s|\theta^r) U(c_n^m(\theta^r)) \geqslant A_n^m(\theta^r),$$

with $b_n^m(\theta^t)$ and $\{P^m(\theta^r)\}_{r\in[t,\infty)}$ given, for all histories θ^r and states (θ^r, θ_{r+1}) with $r \in [t, \infty)$.

Lemma 1 *The redefined problem (RIA^F) has a unique maximum solution.*

Proof. *Prove by contradiction. Suppose there are two different optimal solutions to problem (RIA^F)— $\left\{c_{n,1}^{m,D}(\theta^r)\right\}_{r\in[t,\infty)}$ and $\left\{c_{n,2}^{m,D}(\theta^r)\right\}_{s\in[t,\infty)}$. Create another consumption allocation $\left\{c_{n,3}^{m,D}(\theta^r)\right\}_{r\in[r,\infty)}$ as a linear combination of the above two solutions, i.e.,*

$$c_{n,3}^{m,D}(\theta^r) = \lambda c_{n,1}^{m,D}(\theta^r) + (1-\lambda) c_{n,2}^{m,D}(\theta^r), \forall r$$

for any $\lambda \in (0,1)$. Thus, $\left\{c_{n,3}^{m,D}(\theta^r)\right\}_{r\in[t,\infty)}$ is both affordable and individual rational, since strictly concave utility function implies

$$\sum_{s=r}^{\infty} \beta^{s-r} \sum_{\theta^s|\theta^r} \pi(\theta^s|\theta^r) U(c_{n,3}^{m,D}(\theta^s)) > \sum_{s=r}^{\infty} \beta^{s-r} \sum_{\theta^s|\theta^r} \pi(\theta^s|\theta^r) U(c_{n,1(2)}^{m,D}(\theta^s)) \geqslant A_n^m(\theta^r),$$

for all histories θ^r with $r \in [t, \infty)$. But $\left\{c_{n,3}^{m,D}(\theta^r)\right\}_{r\in[t,\infty)}$ makes a resident strictly better off at the first place, if it sets $r = t$, which contradicts with the assumption that $\left\{c_{n,1}^{m,D}(\theta^r)\right\}_{r\in[t,\infty)}$ and $\left\{c_{n,2}^{m,D}(\theta^r)\right\}_{t\in[t,\infty)}$ are the optimal solutions. ■

STEP 2: For $F_n^m(\theta^t) \in \mathbb{R}$, define another optimization problem (RP$^{\mathrm{F}}$) as an augmented version of (RIA$^{\mathrm{F}}$)

$$W_n^{m,F}(\theta^t, b_n^m(\theta^t), F(\theta^t)) \equiv \max_{\{c_n^m(\theta^r)\}_{r\in[t,\infty)}} \sum_{r=t}^{\infty} \beta^{r-t} \sum_{\theta^r|\theta^t} \pi(\theta^r|\theta^t) U(c_n^m(\theta^r)), \tag{RP$^{\mathrm{F}}$}$$

subject to the summation of all future budget constraints after history θ^t being discounted to period 0,

$$\sum_{r=t}^{\infty} \sum_{\theta^r|\theta^t} P^m(\theta^r) c_n^m(\theta^r) = \sum_{r=t}^{\infty} \sum_{\theta^r|\theta^t} P^m(\theta^r) e_n^m(\theta^r) + P^m(\theta^t) b_n^m(\theta^t) + F_n^m(\theta^t),$$

and the participation constraint in the domestic asset market,

$$\sum_{s=r}^{\infty} \beta^{s-r} \sum_{\theta^s|\theta^r} \pi(\theta^s|\theta^r) U(c_n^m(\theta^s)) \geqslant A_n^m(\theta^r),$$

with $b_n^m(\theta^t), F_n^m(\theta^t)$ and $\{P^m(\theta^r)\}_{r\in[t,\infty)}$ given, for all histories θ^r with $r \in [t, \infty)$.

Consider all histories θ^t and initial bond holdings $b_n^m(\theta^t)$. By definition, the value function of problem (RIA$^{\mathrm{F}}$) equals the value function of problem (RP$^{\mathrm{F}}$) given $F_n^m(\theta^t) = 0$.

$$V_n^m(\theta^t, b_n^m(\theta^t)) = W_n^{m,F}(\theta^t, b_n^m(\theta^t), 0),$$

Define

$$F_n^m(\theta^t) = \sum_{r=t}^{\infty} \sum_{\theta^r|\theta^t} P^m(\theta^r) \left[f_n^m(\theta^r) - \sum_{\theta_{r+1}} q(\theta^r, \theta_{r+1}) f_n^m(\theta^r, \theta_{r+1}) \right].$$

Then, the continuation value of the original resident's problem (RP) at θ^t equals the value function of the newly defined problem (RP$^{\mathrm{F}}$).

$$\sum_{r=t}^{\infty} \beta^{r-t} \sum_{\theta^r|\theta^t} \pi(\theta^r|\theta^t) U(c_n^m(\theta^r)) = W_n^{m,F}(\theta^t, b_n^m(\theta^t), F_n^m(\theta^t)).$$

But, this equality is true only if the international participation constraint (5) in the problem (RP) are also satisfied in the problem (RP$^{\mathrm{F}}$).

$$\sum_{r=t}^{\infty} \beta^{r-t} \sum_{\theta^r|\theta^t} \pi(\theta^r|\theta^t) U(c_n^m(\theta^r)) \geqslant V_n^m(\theta^t, b_n^m(\theta^t)).$$

Substitute both sides of the above constraint for value functions in (RP$^{\mathrm{F}}$) to obtain

$$W_n^{m,F}(\theta^t, b_n^m(\theta^t), F_n^m(\theta^t)) \geqslant W_n^{m,F}(\theta^t, b_n^m(\theta^t), 0).$$

Since $W_n^{m,F}(\theta^t, b_n^m(\theta^t), \cdot)$ is strictly increasing in $F_n^m(\theta^t)$, the above inequality further implies

$$F_n^m(\theta^t) \geqslant 0.$$

Moreover, if Eq. (5) holds with equality, then $F_n^m(\theta^r) = 0$. By now, the reasoning suffices to prove the following Lemma 2.

Lemma 2 *For all n, m, and θ^t with $t \in [0, \infty)$, the international participation constraint (5) at θ^t implies*

$$\sum_{r=t}^{\infty} \sum_{\theta^r|\theta^t} P^m(\theta^r) \left[f_n^m(\theta^r) - \sum_{\theta_{r+1}} q(\theta^r, \theta_{r+1}) f_n^m(\theta^r, \theta_{r+1}) \right] \geqslant 0.$$

Moreover, if Eq. (5) holds with equality at θ^t, *then*

$$\sum_{r=t}^{\infty}\sum_{\theta^r|\theta^t} P^m(\theta^r)\left[f_n^m(\theta^r)-\sum_{\theta_{r+1}} q(\theta^r,\theta_{r+1})f_n^m(\theta^r,\theta_{r+1})\right]=0.$$

STEP 3: Now, it is ready to prove Proposition 1. For some n, m, and θ^t with $t\in[0,\infty)$, $\mu_n^m(\theta^t)>0$ implies the international participation constraint of type n residents in country m holds with equality at θ^t. Lemma 2 concludes $F_n^m(\theta^t)=0$. By definition, the consumption allocation, $\left\{c_n^{m,D}(\theta^r)\right\}_{r\in[t,\infty)}$, solves the problem (RIA) started at θ^t, and $\{c_n^m(\theta^r)\}_{r\in[0,\infty)}$ solves the problem (RP) at period 0. That is to say, $\left\{c_n^{m,D}(\theta^r)\right\}_{r\in[t,\infty)}$ solves $W_n^{m,F}(\theta^t,b_n^m(\theta^t),0)$ and $\{c_n^m(\theta^r)\}_{r\in[t,\infty)}\subseteq\{c_n^m(\theta^r)\}_{r\in[0,\infty)}$ solves $W_n^{m,F}(\theta^t,b_n^m(\theta^t),F_n^m(\theta^t))$. Since $F_n^m(\theta^t)=0$, these two optimal allocations both solve $W_n^{m,F}(\theta^t,b_n^m(\theta^t),0)$, i.e., they both solve the Resident's International Autarky problem (RIA) and its redefined problem (RIA$^{\mathrm{F}}$) with value function $V_n^m(\theta^t,b_n^m(\theta^t))$. Finally, by Lemma 1, the optimization problem (RIA$^{\mathrm{F}}$) has a unique solution that proves $c_n^{m,D}(\theta^r)$ and $c_n^m(\theta^r)$ are identical at all histories θ^r with $r\in[t,\infty)$.

A.3 Proof of Proposition 2

In equilibrium, bond prices are determined by (A.5) and (A.6) for all histories (θ^t,θ_{t+1}) with $t\in[0,\infty)$.

$$\begin{cases} p(\theta^t,\theta_{t+1})=\beta\frac{U'(c(\theta^t,\theta_{t+1}))}{U'(c(\theta^t))}\pi(\theta_{t+1}|\theta^t)\frac{1+A_2-\left(1+\nu(\theta^t,\theta_{t+1})\right)A_1}{1+A_3}; \\ q(\theta^t,\theta_{t+1})=\beta\frac{U'(c(\theta^t,\theta_{t+1}))}{U'(c(\theta^t))}\pi(\theta_{t+1}|\theta^t)\frac{1+A_2}{1+A_3}, \end{cases}$$

where

$$\begin{aligned} A_1 &= \mu(\theta^t,\theta_{t+1})\beta^{-t-1}\frac{U'(c^D(\theta^t,\theta_{t+1}))}{U'(c(\theta^t,\theta_{t+1}))}\frac{1}{\pi(\theta^t,\theta_{t+1})}; \\ A_2 &= \sum_{s=0}^{t+1}\sum_{\theta^t,\theta_{t+1}|\theta^s}\mu(\theta^s)\beta^{-s}\frac{\pi(\theta^t,\theta_{t+1}|\theta^s)}{\pi(\theta^t,\theta_{t+1})}; \\ A_3 &= \sum_{s=0}^{t}\sum_{\theta^t|\theta^s}\mu(\theta^s)\beta^{-s}\frac{\pi(\theta^t|\theta^s)}{\pi(\theta^t)}. \end{aligned}$$

The proof shows in three steps how the interaction between international and domestic bond prices makes the different types reach their upper limits of international borrowing concurrently.

STEP 1: Consider the international bond pricing rule (A.6). The Lagrangian multipliers imposed on the international participation constraints must be non-negative.

$$\mu(\theta^t,\theta_{t+1})\geqslant 0\Rightarrow$$

$$A_2 = A_3 + \mu(\theta^t, \theta_{t+1})\beta^{-t-1}\frac{1}{\pi(\theta^t, \theta_{t+1})} \geqslant A_3,$$

and

$$A_2 = A_3 \text{ if } \mu(\theta^t, \theta_{t+1}) = 0.$$

Therefore, the international bond price represents the highest MRS among all types in all countries.

$$q(\theta^t, \theta_{t+1}) \geqslant \beta\frac{U'(c(\theta^t, \theta_{t+1}))}{U'(c(\theta^t))}\pi(\theta_{t+1}|\theta^t), \text{ with equality iif } \mu_n^m(\theta^t, \theta_{t+1}) = 0. \tag{A.7}$$

Step 2: Consider the domestic bond pricing rule (A.5) for any country. Substitute all A's and rearrange.

$$\begin{aligned} p(\theta^t, \theta_{t+1}) &= \beta\frac{U'(c(\theta^t, \theta_{t+1}))}{U'(c(\theta^t))}\pi(\theta_{t+1}|\theta^t) \times \\ & \left[1 + \frac{(1 - \frac{U'(c^D(\theta^t,\theta_{t+1}))}{U'(c(\theta^t,\theta_{t+1}))})\frac{\mu(\theta^t,\theta_{t+1})\beta^{-t-1}}{\pi(\theta^t,\theta_{t+1})} - \frac{U'(c^D(\theta^t,\theta_{t+1}))}{U'(c(\theta^t,\theta_{t+1}))}\frac{\nu(\theta^t,\theta_{t+1})\mu(\theta^t,\theta_{t+1})\beta^{-t-1}}{\pi((\theta^t,\theta_{t+1}))}}{1 + \sum_{s=0}^t \sum_{\theta^t|\theta^s} \mu(\theta^s)\beta^{-s}\frac{\pi(\theta^t|\theta^s)}{\pi(\theta^t)}}\right]. \end{aligned}$$

For some type n in m, if one observes $\mu_n^m(\theta^t, \theta_{t+1}) = 0$, then the domestic bond price in m is determined by n's MRS.

$$p^m(\theta^t, \theta_{t+1}) = \beta\frac{U'(c_n^m(\theta^t, \theta_{t+1}))}{U'(c_n^m(\theta^t))}\pi(\theta_{t+1}|\theta^t).$$

If one observes $\mu_n^m(\theta^t, \theta_{t+1}) > 0$ instead, then $c_n^{m,D}(\theta^t, \theta_{t+1}) = c_n^m(\theta^t, \theta_{t+1})$ by Proposition 1. The relationship between domestic bond price in m and its type n's MRS is

$$p^m(\theta^t, \theta_{t+1}) = \beta\frac{U'(c_n^m(\theta^t, \theta_{t+1}))}{U'(c_n^m(\theta^t))}\pi(\theta_{t+1}|\theta^t)\left[1 - \frac{\frac{\nu_n^m(\theta^t,\theta_{t+1})\mu_n^m(\theta^t,\theta_{t+1})\beta^{-t-1}}{\pi((\theta^t,\theta_{t+1}))}}{1 + \sum_{s=0}^t \sum_{\theta^t|\theta^s} \mu_n^m(\theta^s)\beta^{-s}\frac{\pi(\theta^t|\theta^s)}{\pi(\theta^t)}}\right]. \tag{A.8}$$

The Lagrangian multipliers imposed on domestic participation constraints in problem (RIA) must be non-negative.

$$\begin{gathered} v_n^m(\theta^t, \theta_{t+1}) \geqslant 0 \Rightarrow \\ p^m(\theta^t, \theta_{t+1}) \leqslant \beta\frac{U'(c_n^m(\theta^t, \theta_{t+1}))}{U'(c_n^m(\theta^t))}\pi(\theta_{t+1}|\theta^t), \text{ with equality if } v_n^m(\theta^t, \theta_{t+1}) = 0. \end{gathered} \tag{A.9}$$

Step 3: If $\mu_n^m(\theta^t, \theta_{t+1}) > 0$ for some type n residents in country m, then Eqs. (A.7) and (A.9) together ensure the international bond price is strictly greater than the domestic bond price in m.

$$q(\theta^t, \theta_{t+1}) > \beta\frac{U'(c_n^m(\theta^t, \theta_{t+1}))}{U'(c_n^m(\theta^t))} \geqslant p^m(\theta^t, \theta_{t+1}) \Rightarrow$$

$$q(\theta^t, \theta_{t+1}) > p^m(\theta^t, \theta_{t+1}).$$

This strict inequality, the other way around, implies that multipliers $\mu_n^m(\theta^t, \theta_{t+1})$ are positive for all types in m as well.

$$q(\theta^t, \theta_{t+1}) > p^m(\theta^t, \theta_{t+1}) \Rightarrow$$

$$\beta \frac{U'(c_n^m(\theta^t, \theta_{t+1}))}{U'(c_n^m(\theta^t))} \pi(\theta_{t+1}|\theta^t) \frac{1 + A_{n,2}^m}{1 + A_{n,3}^m} > \beta \frac{U'(c_n^m(\theta^t, \theta_{t+1}))}{U'(c_n^m(\theta^t))} \pi(\theta_{t+1}|\theta^t) \frac{1 + A_{n,2}^m - \left(1 + \nu_n^m(\theta^t, \theta_{t+1})\right) A_{n,1}^m}{1 + A_{n,3}^m} \Rightarrow$$

$$\left(1 + \nu_n^m(\theta^t, \theta_{t+1})\right) \mu_n^m(\theta^t, \theta_{t+1}) \beta^{-r-1} \frac{1}{\pi(\theta^t, \theta_{t+1}|\theta^t)} > 0 \Rightarrow$$

$$\left(1 + \nu_n^m(\theta^t, \theta_{t+1})\right) \mu_n^m(\theta^t, \theta_{t+1}) > 0.$$

Since $\nu_n^m(\theta^t, \theta_{t+1}) \geqslant 0$ for all n in m,

$$\mu_n^m(\theta^t, \theta_{t+1}) > 0, \text{ for all } n, m.$$

A.4 Proof of Proposition 3

Given $\mu_n^m(\theta^t) > 0$ for some n, m at θ^t with $t \in [0, \infty)$, the international participation constraint (5) in the resident's problem (RP) is binding at θ^t.

$$\sum_{r=t}^{\infty} \beta^{r-t} \sum_{\theta^r|\theta^t} \pi(\theta^r|\theta^t) U(c_n^m(\theta^r)) = V_n^m(\theta^t, b_n^m(\theta^t)),$$

where $c_n^m(\theta^r)$ and $b_n^m(\theta^t)$ are the optimal consumption at θ^r and the optimal domestic bond holdings at θ^t in problem (RP). The value function $V_n^m(\theta^t, b_n^m(\theta^t))$ of problem (RIA) started at θ^t affects the optimal consumption allocation after θ^t in problem (RP) through this binding constraint. In addition, $\nu_n^m(\theta^r) > 0$ for the same n and m above with $r \in [t, \infty)$ implies the domestic participation constraint in the Resident's International Autarky problem (RIA) binds at θ^r.

$$\sum_{s=r}^{\infty} \beta^{s-r} \sum_{\theta^s|\theta^r} \pi(\theta^s|\theta^r) U(c_n^{m,D}(\theta^s, \theta^t, b_n^m(\theta^t))) = A_n^m(\theta^r),$$

where $c_n^{m,D}(\theta^s, \theta^t, b_n^m(\theta^t))$ denotes the optimal consumption at θ^s in problem (RIA) started at θ^t with initial domestic bond holdings, $b_n^m(\theta^t)$. This equation implicitly defines $b_n^m(\theta^t) = \overline{B}_n^m(\theta^t)$.

A.5 Proof of Proposition 4

By Lemma 2 and Proposition 3, the international participation constraint (5) at θ^t with $t \in [0, \infty)$ implies the following set of inequality and equality constraints

$$\begin{cases} \sum_{r=t}^{\infty} \sum_{\theta^r | \theta^t} P^m(\theta^r) \left[f_n^m(\theta^r) - \sum_{\theta_{r+1}} q(\theta^r, \theta_{r+1}) f_n^m(\theta^r, \theta_{r+1}) \right] \geqslant 0; \\ b_n^m(\theta^t, \theta_{t+1}) = \overline{B}_n^m(\theta^t, \theta_{t+1}) \text{ if } \mu_n^m(\theta^t, \theta_{t+1}) > 0 \text{ and} \\ \nu_n^m(\theta^r, \theta_{r+1}) > 0 \text{ for some } \theta^r \text{ with } r \in [t, \infty). \end{cases} \tag{A.10}$$

The proof proceeds in the following three steps: define, solve and compare.

Step 1: For all histories θ^t with $t \in [0, \infty)$, replacing Eq. (5) in problem (RP) with weaker constraints (A.10) creates an alternative (convex) resident's problem (RP$^{\text{a}}$).

$$\max_{\{c_n^m(\theta^t), b_n^m(\theta^t, \theta_{t+1}), f_n^m(\theta^t, \theta_{t+1})\}_{t \in [0,\infty)}} \sum_{t=0}^{\infty} \beta^t \sum_{\theta^t} \pi(\theta^t) U(c_n^m(\theta^t)), \tag{RP$^{\text{a}}$}$$

subject to the budget constraint,

$$\begin{aligned} & e_n^m(\theta^t) + b_n^m(\theta^t) + f_n^m(\theta^t) \\ \geqslant \; & c_n^m(\theta^t) + \sum_{\theta_{t+1}} p^m(\theta^t, \theta_{t+1}) b_n^m(\theta^t, \theta_{t+1}) + \sum_{\theta_{t+1}} q(\theta^t, \theta_{t+1}) f_n^m(\theta^t, \theta_{t+1}), \end{aligned} \tag{A.11}$$

the weaker version of participation constraint in the international asset market,

$$\sum_{r=t}^{\infty} \sum_{\theta^r | \theta^t} P^m(\theta^r) \left[f_n^m(\theta^r) - \sum_{\theta_{r+1}} q(\theta^r, \theta_{r+1}) f_n^m(\theta^r, \theta_{r+1}) \right] \geqslant 0 \tag{A.12}$$

and

$$b_n^m(\theta^t, \theta_{t+1}) = \overline{B}_n^m(\theta^t, \theta_{t+1}) \text{ if } \mu_n^m(\theta^t, \theta_{t+1}) > 0 \text{ and } \nu_n^m(\theta^r, \theta_{r+1}) > 0 \text{ for some } \theta^r, \tag{A.13}$$

the no-Ponzi conditions,

$$b_n^m(\theta^t, \theta_{t+1}) \geqslant -\overline{B}, f_n^m(\theta^t, \theta_{t+1}) \geqslant -\overline{F},$$

with initial bond holdings,

$$b_n^m(\theta^0) \text{ and } f_n^m(\theta^0),$$

and the price sequence,

$$\{p^m(\theta^t, \theta_{t+1}), q(\theta^t, \theta_{t+1})\}_{t \in [0,\infty)} \text{ given},$$

for all histories θ^t and all states (θ^t, θ_{t+1}) with $t \in [0, \infty)$.

Let $\kappa\left(\theta^t\right), \mu_f\left(\theta^t\right)$ and $\mu_b\left(\theta^t\right)$ be the Lagrangian multipliers on the budget constraint (A.11), the non-negative foreign capital inflow condition (A.12), and the domestic bond holding restriction (A.13), respectively. First order conditions are, with respect to $c(\theta^t)$,

$$\kappa(\theta^t) = \beta^t \pi(\theta^t) U'(c(\theta^t));$$

with respect to $b(\theta^t, \theta_{t+1})$,

$$p(\theta^t, \theta_{t+1})\kappa(\theta^t) = \begin{cases} \kappa(\theta^t, \theta_{t+1}) - \mu_b(\theta^t, \theta_{t+1}) \text{ if } \mu(\theta^t, \theta_{t+1}) > 0, \nu(\theta^r, \theta_{r+1}) > 0; \\ \kappa(\theta^t, \theta_{t+1}) \text{ otherwise;} \end{cases}$$

and with respect to $f(\theta^t, \theta_{t+1})$,

$$q(\theta^t, \theta_{t+1})\kappa(\theta^t) = \kappa(\theta^t, \theta_{t+1}) + \sum_{s=0}^{t+1} \sum_{\theta^t, \theta_{t+1}|\theta^s} \mu_f(\theta^s) P(\theta^t, \theta_{t+1}) - \sum_{s=0}^{r} \sum_{\theta^t|\theta^s} \mu_f(\theta^s) P(\theta^t) q(\theta^t, \theta_{t+1}).$$

Use all of them to generate domestic and international bond pricing rules as follow:

$$p(\theta^t, \theta_{t+1}) = \begin{cases} \beta \frac{U'(c(\theta^t, \theta_{t+1}))}{U'(c(\theta^t))} \pi(\theta_{t+1}|\theta^t)[1 - \frac{\mu_b(\theta^t, \theta_{t+1})}{\kappa(\theta^t, \theta_{t+1})}] \text{ if } \mu(\theta^t, \theta_{t+1}) > 0, \nu(\theta^r, \theta_{r+1}) > 0; \\ \beta \frac{U'(c(\theta^t, \theta_{t+1}))}{U'(c(\theta^t))} \pi(\theta_{t+1}|\theta^t) \text{ otherwise,} \end{cases} \tag{A.14}$$

and

$$q(\theta^t, \theta_{t+1}) = \beta \frac{U'(c(\theta^t, \theta_{t+1}))}{U'(c(\theta^t))} \pi(\theta_{t+1}|\theta^t) \frac{1 + \sum_{s=0}^{t+1} \sum_{\theta^t, \theta_{t+1}|\theta^s} \frac{\mu_f(\theta^s) P(\theta^t, \theta_{t+1})}{\kappa(\theta^t, \theta_{t+1})}}{1 + \sum_{s=0}^{t} \sum_{\theta^t|\theta^s} \frac{\mu_f(\theta^s) P(\theta^t)}{\kappa(\theta^t)}}. \tag{A.15}$$

Step 2: Consider any one country in the world and all histories (θ^t, θ_{t+1}) with $\mu(\theta^t, \theta_{t+1}) > 0$, since non-convexity only becomes problematic when the international participation constraints are binding.

For types of residents with $\mu(\theta^t, \theta_{t+1}) > 0$ and $\nu(\theta^r, \theta_{r+1}) = 0$ at all θ^r with $r \in [t, \infty)$, or n_A types as in the model, the domestic bond pricing rule (A.14) degenerates into

$$p(\theta^t, \theta_{t+1}) = \beta \frac{U'(c(\theta^t, \theta_{t+1}))}{U'(c(\theta^t))} \pi(\theta_{t+1}|\theta^t) = \frac{\kappa(\theta^t, \theta_{t+1})}{\kappa(\theta^t)}. \tag{A.16}$$

The Arrow-Debreu price of a $(t+1)$-period matured domestic bond at period 0 can be, therefore, written as

$$P(\theta^t, \theta_{r+1}) = P(\theta^t) p(\theta^t, \theta_{t+1}) = P(\theta^t) \frac{\kappa(\theta^t, \theta_{t+1})}{\kappa(\theta^t)}.$$

Plug it into the international bond pricing rule (A.15) to obtain

$$q(\theta^t, \theta_{t+1}) = \beta \frac{U'(c(\theta^t, \theta_{t+1}))}{U'(c(\theta^t))} \pi(\theta_{t+1}|\theta^t) \frac{1 + \sum_{s=0}^{t+1} \sum_{\theta^t, \theta_{t+1}|\theta^s} \frac{\mu_f(\theta^s) P(\theta^t)}{\kappa(\theta^t)}}{1 + \sum_{s=0}^{t} \sum_{\theta^t|\theta^s} \frac{\mu_f(\theta^s) P(\theta^t)}{\kappa(\theta^t)}}.$$

To rescale the Lagrangian multipliers $\mu_f(\theta^s)$, define

$$\mu'_f(\theta^s) = \mu_f(\theta^s)\beta^s \frac{P(\theta^t)\pi(\theta^t, \theta_{t+1})}{\kappa(\theta^t)\pi(\theta^t, \theta_{t+1}|\theta^s)}, \tag{A.17}$$

where $\kappa(\theta^t) > 0$, since the budget constraint always binds in equilibrium and notice

$$\frac{\pi(\theta^t, \theta_{t+1}|\theta^s)}{\pi(\theta^t, \theta_{t+1})} = \frac{\pi(\theta_{t+1}|\theta^t)\pi(\theta^t|\theta^s)}{\pi(\theta_{t+1}|\theta^t)\pi(\theta^t)} = \frac{\pi(\theta^t|\theta^s)}{\pi(\theta^t)}.$$

Using the definition in Eq. (A.17) to replace $\mu_f(\theta^s)$ yields

$$q(\theta^t, \theta_{t+1}) = \beta \frac{U'(c(\theta^t, \theta_{t+1}))}{U'(c(\theta^t))} \pi(\theta_{t+1}|\theta^t) \frac{1 + \sum_{s=0}^{t+1} \sum_{\theta^t, \theta_{t+1}|\theta^s} \mu'_f(\theta^s)\beta^{-s} \frac{\pi(\theta^t, \theta_{t+1}|\theta^s)}{\pi(\theta^t, \theta_{t+1})}}{1 + \sum_{s=0}^{t} \sum_{\theta^t|\theta^s} \mu'_f(\theta^s)\beta^{-s} \frac{\pi(\theta^t|\theta^s)}{\pi(\theta^t)}}. \tag{A.18}$$

For types of residents in the same country with $\mu(\theta^t, \theta_{t+1}) > 0$ and $\nu(\theta^r, \theta_{r+1}) > 0$ at some θ^r with $r \in [t, \infty)$, or n_B types as in the model, they must confront with the same international and domestic bond prices as determined above. To rescale the Lagrangian multipliers $\mu_b(\theta^t, \theta_{t+1})$, define

$$\mu'_b(\theta^t, \theta_{t+1}) \equiv \frac{\mu_b(\theta^t, \theta_{t+1})}{U'(c(\theta^t, \theta_{t+1}))\mu'_f(\theta^t, \theta_{t+1})\beta^{-t}} \left[1 + \sum_{s=0}^{t} \sum_{\theta^t|\theta^s} \mu'_f(\theta^s)\beta^{-s} \frac{\pi(\theta^t|\theta^s)}{\pi(\theta^t)}\right], \tag{A.19}$$

where $\mu'_f(\theta^t, \theta_{t+1}) > 0$, since the non-negative foreign capital inflow condition always binds in equilibrium. Then, the domestic bond pricing rule (A.14) for types n_B suggests

$$p(\theta^t, \theta_{t+1}) = \beta \frac{U'(c(\theta^t, \theta_{t+1}))}{U'(c(\theta^t))} \pi(\theta_{t+1}|\theta^t)[1 - \frac{\mu_b(\theta^t, \theta_{t+1})}{\kappa(\theta^t, \theta_{t+1})}].$$

Using the definition in Eq. (A.19) to replace $\mu_b(\theta^r, \theta_{r+1})$ generates

$$\begin{aligned} p(\theta^t, \theta_{t+1}) \quad &= \quad \beta \frac{U'(c(\theta^t, \theta_{t+1}))}{U'(c(\theta^t))} \pi(\theta_{t+1}|\theta^t) \times \\ & \left[1 - \frac{\mu'_b(\theta^t, \theta_{t+1})\mu'_f(\theta^t, \theta_{t+1})\beta^{-t-1} \frac{1}{\pi(\theta^t, \theta_{t+1})}}{1 + \sum_{s=0}^{t} \sum_{\theta^t|\theta^s} \mu'_f(\theta^s)\beta^{-s} \frac{\pi(\theta^t|\theta^s)}{\pi(\theta^t)}}\right] \\ \text{if } \mu(\theta^t, \theta_{t+1}) \quad &> \quad 0, \nu(\theta^r, \theta_{r+1}) > 0. \end{aligned} \tag{A.20}$$

Step 3: Now, recall bond price rules (A.6) and (A.5) in the Trade Equilibrium with all A's substituted out.

$$\begin{aligned} q(\theta^t, \theta_{t+1}) \quad &= \quad \beta \frac{U'(c(\theta^t, \theta_{t+1}))}{U'(c(\theta^t))} \pi(\theta_{t+1}|\theta^t) \frac{1 + A_2}{1 + A_3} \\ &= \quad \beta \frac{U'(c(\theta^t, \theta_{t+1}))}{U'(c(\theta^t))} \pi(\theta_{t+1}|\theta^t) \frac{1 + \sum_{s=0}^{t+1} \sum_{\theta^t, \theta_{t+1}|\theta^s} \mu(\theta^s)\beta^{-s} \frac{\pi(\theta^t, \theta_{t+1}|\theta^s)}{\pi(\theta^t, \theta_{t+1})}}{1 + \sum_{s=0}^{t} \sum_{\theta^t|\theta^s} \mu(\theta^s)\beta^{-s} \frac{\pi(\theta^t|\theta^s)}{\pi(\theta^t)}}, \end{aligned} \tag{A.21}$$

and

$$
\begin{aligned}
p(\theta^t, \theta_{t+1}) &= \beta \frac{U'(c(\theta^t, \theta_{t+1}))}{U'(c(\theta^t))} \pi(\theta_{t+1}|\theta^t) \frac{1 + A_2 - \left(1 + \nu(\theta^t, \theta_{t+1})\right) A_1}{1 + A_3} \\
&= \beta \frac{U'(c(\theta^t, \theta_{t+1}))}{U'(c(\theta^t))} \pi(\theta_{t+1}|\theta^t) \times \\
&\quad \left[\frac{1 + \sum_{s=0}^{t+1} \sum_{\theta^t, \theta_{t+1}|\theta^s} \mu(\theta^s) \beta^{-s} \frac{\pi(\theta^t, \theta_{t+1}|\theta^s)}{\pi(\theta^t, \theta_{t+1})} - \frac{\left(1+\nu(\theta^t, \theta_{t+1})\right)\mu(\theta^t, \theta_{t+1})}{\pi(\theta^t, \theta_{t+1})} \beta^{-t-1} \frac{U'(c^D(\theta^t, \theta_{t+1}))}{U'(c(\theta^t, \theta_{t+1}))}}{1 + \sum_{s=0}^{t} \sum_{\theta^t|\theta^s} \mu(\theta^s) \beta^{-s} \frac{\pi(\theta^t|\theta^s)}{\pi(\theta^t)}} \right] \\
&= \begin{cases} \beta \frac{U'(c(\theta^t, \theta_{t+1}))}{U'(c(\theta^t))} \pi(\theta_{t+1}|\theta^t) \times \\ \left[1 - \frac{\nu(\theta^t, \theta_{r+1}) \mu(\theta^t, \theta_{t+1}) \beta^{-t-1} \frac{1}{\pi(\theta^t, \theta_{t+1})}}{1 + \sum_{s=0}^{t} \sum_{\theta^t|\theta^s} \mu(\theta^s) \beta^{-s} \frac{\pi(\theta^t|\theta^s)}{\pi(\theta^t)}} \right] & \text{if } \mu(\theta^t, \theta_{t+1}) > 0, \nu(\theta^r, \theta_{r+1}) > 0; \\ \beta \frac{U'(c(\theta^t, \theta_{t+1}))}{U'(c(\theta^t))} \pi(\theta_{t+1}|\theta^t) \text{ otherwise.} \end{cases}
\end{aligned}
\tag{A.22}
$$

Notice the domestic bond pricing rules (A.20) plus (A.16) and international bond pricing rule (A.18) in the alternative (convex) resident's problem (RP^a) are, respectively, identical to the corresponding (A.22) and (A.21) from the original (non-convex) resident's problem (RP). Both maximization problems have identical first order conditions; hence, the same optimal solution. An alternative maximization problem has been defined with the same objective function and a convex constraint set that is a super set of the original (non-convex in general) constraint set. The optimal solution is a global maximum for problem (RP^a) because of convexity. Therefore, it must be the global maximum for the original problem (RP), which has the same objective function and first order conditions as in problem (RP^a), except for a smaller constraint set. This proves the sufficiency of first order conditions for a global max in the original problem (RP).

A.6 Proof of Proposition 5

Like the proposition itself, the proof has been divided into three parts:

Part 1: For the analysis below, consider any history (θ^t, θ_{t+1}) with $t \in [0, \infty)$. Proposition 5(I) can be easily read from inequality (A.7) in the proof of Proposition 2.

$$
q(\theta^t, \theta_{t+1}) \geqslant \beta \frac{U'(c_n^m(\theta^t, \theta_{t+1}))}{U'(c_n^m(\theta^t))} \pi(\theta_{t+1}|\theta^t), \text{ for all } n, m. \tag{A.23}
$$

Part 2: Without loss of generality, suppose residents with the highest MRS across the world live in some countries m as some types n, where m and n are subsets of $\{1, 2, ..., M\}$ and $\{1, 2, ..., N\}$, respectively.

Then, their MRS determines the international bond price.

$$q(\theta^t, \theta_{t+1}) = \beta \frac{U'(c_n^m(\theta^t, \theta_{t+1}))}{U'(c_n^m(\theta^t))} \pi(\theta_{t+1}|\theta^t) \text{ and } \mu_n^m(\theta^t, \theta_{t+1}) = 0.$$

By Proposition 2, $\mu_n^m(\theta^t, \theta_{t+1}) = 0$ implies that $\mu_{n^-}^m(\theta^t, \theta_{t+1}) = 0$ for all other types in m, where $n^- = \{1, 2, ..., N\} \setminus n$. Eq. (A.8) in the proof of Proposition 2 states

$$p^m(\theta^t, \theta_{t+1}) = \beta \frac{U'(c_n^m(\theta^t, \theta_{t+1}))}{U'(c_n^m(\theta^t))} \pi(\theta_{t+1}|\theta^t) \text{ for all } n \text{ in } m.$$

The MRS is equalized within country m. Therefore,

$$p^m(\theta^t, \theta_{t+1}) = q(\theta^t, \theta_{t+1}) \text{ for all } m. \tag{A.24}$$

All other countries $m^- = \{1, 2, ..., M\} \setminus m$ which are internationally participation constrained as a whole have

$$q(\theta^t, \theta_{t+1}) < \beta \frac{U'(c_n^{m^-}(\theta^t, \theta_{t+1}))}{U'(c_n^{m^-}(\theta^t))} \pi(\theta_{t+1}|\theta^t) \text{ and } \mu_n^{m^-}(\theta^t, \theta_{t+1}) > 0, \text{ for all } n \text{ in } m^-.$$

Inequality (A.9) in the proof of Proposition 2 yields

$$p^{m^-}(\theta^t, \theta_{t+1}) \leqslant \beta \frac{U'(c_n^{m^-}(\theta^t, \theta_{t+1}))}{U'(c_n^{m^-}(\theta^t))} \pi(\theta_{t+1}|\theta^t) \text{ for all } n \text{ in } m^-. \tag{A.25}$$

For all countries, combine Eq. (A.24) and inequality (A.25) to obtain

$$p^m(\theta^t, \theta_{t+1}) = \begin{cases} \leqslant \beta \frac{U'(c_n^m(\theta^t, \theta_{t+1}))}{U'(c_n^m(\theta^t))} \pi(\theta_{t+1}|\theta^t) \text{ if } \mu_n^m(\theta^t, \theta_{t+1}) > 0; \\ = q(\theta^t, \theta_{t+1}) \text{ if } \mu_n^m(\theta^t, \theta_{t+1}) = 0, \end{cases}$$

or more general like Proposition 5(II),

$$p^m(\theta^t, \theta_{t+1}) \leqslant \beta \frac{U'(c_n^m(\theta^t, \theta_{t+1}))}{U'(c_n^m(\theta^t))} \pi(\theta_{t+1}|\theta^t), \text{ for all } n, m. \tag{A.26}$$

Part 3: Connecting inequality (A.23) and (A.26) in one direction,

$$p^m(\theta^t, \theta_{t+1}) \leqslant \beta \frac{U'(c_n^m(\theta^t, \theta_{t+1}))}{U'(c_n^m(\theta^t))} \pi(\theta_{t+1}|\theta^t) \leqslant q(\theta^t, \theta_{t+1}),$$

or more directly,

$$p^m(\theta^r, \theta_{r+1}) \leqslant q(\theta^r, \theta_{r+1}), \text{ for all } m,$$

which proves Proposition 5(III).

A.7 Proof of Proposition 6

STEP 1: First, assume bond price sequence $\left\{p^m(\theta^t, \theta_{t+1}), q(\theta^t, \theta_{t+1})\right\}_{t\in[0,\infty)}$ in problems (RP) and (RP^{J}) are identical. It will be shown later they are indeed the same in the two maximization problem's corresponding equilibria. Consider any type n residents in country m at θ^t with initial domestic bond holdings $b_n^m\left(\theta^t\right)$ and bond prices $\{p^m(\theta^r, \theta_{r+1}), q(\theta^r, \theta_{r+1})\}_{r\in[t,\infty)}$ given. The constraint set of the post-default maximization problem (RIA) is a subset of problem (RIA^{J})'s constraint set, since the former set contains constraint (2) to prevent domestic debt default. Therefore, the optimal solution to the (RIA) problem is always feasible in the (RIA^{J}) problem, which proves the following

$$V_n^{m,J}(\theta^t, b_n^m(\theta^t)) \geqslant V_n^m(\theta^t, b_n^m(\theta^t)),\ \text{for all } \theta^t, b_n^m(\theta^t).$$

Using the same argument in the other direction, that is, noting the constraint set of (RP^{J}) is a subset of (RP)'s constraint set proves the result that adding the limited commitment problem on domestic debt makes every type in all countries weakly better.

Maximization problems (RP) and (RP^{J}) will have the same consumption allocation if

$$V_n^{m,J}(\theta^t, b_n^m(\theta^t)) = V_n^m(\theta^t, b_n^m(\theta^t)),\ \text{for all } \theta^t, b_n^m(\theta^t).$$

For them to be equal in all histories, the domestic participation constraint (2) must be slack at all future histories θ^r with $r \in [t, \infty)$ in problem (RIA) started at any present history θ^t with $t \in [0, \infty)$. In other words, the optimal consumption path for n_A types residents with $\nu_n^m(\theta^r) = 0$ for all θ^r stays unchanged in the above two resident's problems. And from the domestic bond pricing rule (A.5), $p^m(\theta^t, \theta_{t+1})$ is determined by the MRS of type n_A residents. As a result, the domestic price sequences in the two problems are identical and so are the international bond price sequences, since international price is the maximum among all domestic prices.

STEP 2: Now, prove the main result, that by adding domestic enforcement problem strictly improves type n_B's utility. Remember, type n_B residents are defined as those who are indifferent between repaying and reneging domestic debt in Resident's International Autarky. By assumption,

$$\frac{U'\left(c_n^m(\theta^t, \theta_{t+1})\right)}{U'\left(c_n^m\left(\theta^t\right)\right)} \neq \frac{U'\left(c_1^m(\theta^t, \theta_{t+1})\right)}{U'\left(c_1^m\left(\theta^t\right)\right)}. \tag{A.27}$$

Without loss of generality, assume type 1 residents belong to group A of type n_A, then the domestic bond price in m is determined by type 1's MRS

$$p^m(\theta^t, \theta_{t+1}) = \beta\frac{U'(c_1^m(\theta^t, \theta_{t+1}))}{U'(c_1^m(\theta^t))}\pi(\theta_{t+1}|\theta^t).$$

Compare this with Eq. (A.8) and use Eq. (A.27). Therefore,

$$\nu_n^m(\theta^t, \theta_{t+1}), \mu_n^m(\theta^t, \theta_{t+1}) > 0.$$

As a result, type n residents are really type n_B residents whose continuation value after (θ^t, θ_{t+1}) in the resident's problem (RP) equals the continuation value in Resident's International Autarky problem (RIA), since the international participation constraint (5) binds, and again equals to the continuation value in the Resident's Autarky problem (RA), since the domestic participation constraint (2) also binds. With Proposition 2, all types in country m have binding international participation constraints and $q(\theta^t, \theta_{t+1}) > p^m(\theta^t, \theta_{t+1})$, which implies $f_n^{m,J}(\theta^r, \theta_{r+1}) < 0$ for all types in problem ($\mathrm{RP^J}$) because prices remain unchanged. This means two things for the same type n (or to be precise, n_B) above but in problem ($\mathrm{RP^J}$). One, its consumption allocation is neither as in Resident's International Autarky nor Resident's Autarky after (θ^r, θ_{r+1}), and two, the international participation constraint (10) binds because otherwise one can always attain higher utility by borrowing more from foreigners as long as (10) is slack.

Next, recall Proposition 1, the result extends to ($\mathrm{RP^J}$). That is, with a binding international participation constraint, the equilibrium consumption path in problem ($\mathrm{RP^J}$) from (θ^r, θ_{r+1}) onward is identical to the path after international debt default in the ($\mathrm{RIA^J}$) problem. Since the consumption path after (θ^r, θ_{r+1}) in ($\mathrm{RIA^J}$) is not resident autarkic, while it sure is in (RIA), the continuation values are such that $V_n^m(\theta^t, b_n^m(\theta^t)) < V_n^{m,J}(\theta^t, b_n^{m,J}(\theta^t))$ because of a strictly concave objective function.

Thus, type n in problem (RP) without domestic commitment can relax the international participation constraint (5) in history that has a strictly positive Lagrangian multiplier. Thereby, borrow more from outside the country and increase utility to a new level, which is strictly greater than the utility in problem ($\mathrm{RP^J}$) with perfect enforcement on domestic debt.

A.8 Proof of Proposition 7

Step 1: It is difficult to compare directly the aggregate welfare in a private setup with the one in a centralized arrangement, where domestic bond prices are not explicitly stated. So, construct an alternative planner's problem ($\mathrm{PP^a}$) by adding the individual domestic participation constraint below,

$$\sum_{r=t}^{\infty} \beta^{r-t} \sum_{\theta^r|\theta^t} \pi(\theta^r|\theta^t) U(c_n^m(\theta^r)) \geqslant A_n^m(\theta^t), \forall n, \tag{A.28}$$

into the original planner's problem (PP) and use the following international participation constraint instead of Eq. (11)

$$\sum_{n=1}^{N} \varphi_n^m \sum_{r=t}^{\infty} \beta^{r-t} \sum_{\theta^r|\theta^t} \pi(\theta^r|\theta^t) U(c_n^m(\theta^r)) \geqslant V^{m,B}(\theta^t), \tag{A.29}$$

where $V^{m,B}(\theta^t)$ is defined as follow:

$$V^{m,B}(\theta^t) \equiv \max_{\{c_n^m(\theta^r)\}_{r\in[t,\infty)}} \sum_{n=1}^{N} \varphi_n^m \sum_{r=t}^{\infty} \beta^{r-t} \sum_{\theta^r|\theta^t} \pi(\theta^r|\theta^t) U(c_n^m(\theta^r)), \tag{PA$^{\mathrm{a}}$}$$

subject to the resource constraint,

$$\sum_{n=1}^{N} e_n^m(\theta^r) + \sum_{n=1}^{N} b_n^m(\theta^r) \geqslant \sum_{n=1}^{N} c_n^m(\theta^r) + \sum_{n=1}^{N} \sum_{\theta_{t+1}} p^m(\theta^r, \theta_{r+1}) b_n^m(\theta^r, \theta_{r+1}), \tag{A.30}$$

the individual domestic participation constraint,

$$\sum_{s=r}^{\infty} \beta^{s-r} \sum_{\theta^s|\theta^r} \pi(\theta^s|\theta^r) U(c_n^m(\theta^s)) \geqslant A_n^m(\theta^r), \forall n, \tag{A.31}$$

with domestic bond clearing condition at every θ^r,

$$\sum_{n=1}^{N} b_n^m(\theta^r) = 0,$$

and the domestic bond price sequence, $\{p(\theta^r, \theta_{r+1})\}_{r\in[t,\infty)}$ given, for all histories θ^r and all states (θ^r, θ_{r+1}) with $r \in [t, \infty)$.

Percieve the post-default problem (PA$^{\mathrm{a}}$) as the alternative planner ignoring the autarky resource constraint in problem (PA), by assuming it can continue to borrow at the domestic interest rate similar to individuals in a private debt setup. Note with the appropriate distribution of initial bond holdings, problem (RP)'s optimal consumption path can solve its equivalent planner's problem (PP$^{\mathrm{a}}$). The domestic imperfect sharing condition in Proposition 7(II) implies international participation constraint (5) binds at history (θ^t, θ_{t+1}) for all types, and consequently (A.29) binds for country m. Using Proposition 1 in the alternative planner's problem, the consumption stream after θ^{t+1} in all N types' post-default problem (RIA) also solves the equivalent planner's post-default problem (PA$^{\mathrm{a}}$). Then, the assumption in Proposition 7(II) implies

$$\sum_{n=1}^{N} \varphi_n^m V_n^m(\theta^t, b_n^m(\theta^t)) = V^{m,B}(\theta^t) < V^m(\theta^t).$$

Notice $V^{m,B}(\theta^t)$ could be higher than $V^m(\theta^t)$ because of a looser resource constraint (A.30). Setting $b_n^m(\theta^r, \theta_{r+1}) = 0$ for all n and (θ^r, θ_{r+1}) is always feasible, but it could also be lower because of the binding participation constraint (A.31). The ordering is mixed given the little knowledge about the endowment structure or how many types are in group B. Condition (II) requires a large amount of type n_B residents, such that the negative effect from Eq. (A.31) is overwhelming. The final step of proof uses the same logic as in the proof of Proposition 6.

A.9 Derivation of the Zero Net Payment Condition (12)

Without loss of generality, consider country 1 at $t = 0$. As in the proof of Proposition 1, denote by $P(t) = \prod_{r=0}^{t} p(r)$ the initial period Arrow-Debreu price for a domestic contingent bond matured after t periods. Use the one-period domestic bond price (14) to obtain

$$P(t) = \begin{cases} p(pq)^{\frac{t}{2}} \text{ for } t = 2k; \\ (pq)^{\frac{t-1}{2}} \text{ for } t = 2k+1. \end{cases}$$

The collection of Arrow-Debreu prices $\{P(t)\}_{t\in[0,\infty)}$ will look like $\left\{p, pq, p^2q, p^2q^2, p^3q^2, p^3q^3, ...\right\}$. By symmetry, the net payment (positive means capital outflow and negative means capital inflow) from country 1 to 2, denoted by $N(t)$, flips back and forth,

$$N(t) = \begin{cases} x^J - y < 0 \text{ for } t = 2k; \\ y - x^J > 0 \text{ for } t = 2k+1. \end{cases}$$

According to Lemma 2, when discounting all future net payments to period 0 using the corresponding Arrow-Debreu prices, the summation of all present values should equal zero because country 1 is internationally participation constrained at period 0.

$$\sum_{t=0}^{\infty} P(t)N(t) = 0.$$

By substituting the expressions for $P(t)$ and $N(t)$ into the above Eq., the zero capital outflow condition (12) is revealed.

$$\begin{aligned} LHS &= [P(0) + P(2) + P(4) + ...]\,(x^J - y) + [P(1) + P(3) + P(5) + ...]\,(y - x^J) \\ &= (p + p^2q + p^3q^2 + ...)(x^J - y) + (pq + p^2q^2 + p^3q^3 + ...)(y - x^J) \\ &= \left(p\sum_{s=1}^{\infty} p^{s-1}q^{s-1}\right)(x^J - y) + \left(\sum_{s=1}^{\infty} p^s q^s\right)(y - x^J) \\ &= \frac{p(x^J - y)}{1 - pq} + \frac{pq(y - x^J)}{1 - pq} \\ &= \frac{p\left[(x^J - y) + q(y - x^J)\right]}{1 - pq} = RHS = 0. \end{aligned}$$

The same method can be used to derive condition (17) in Section 4.3.

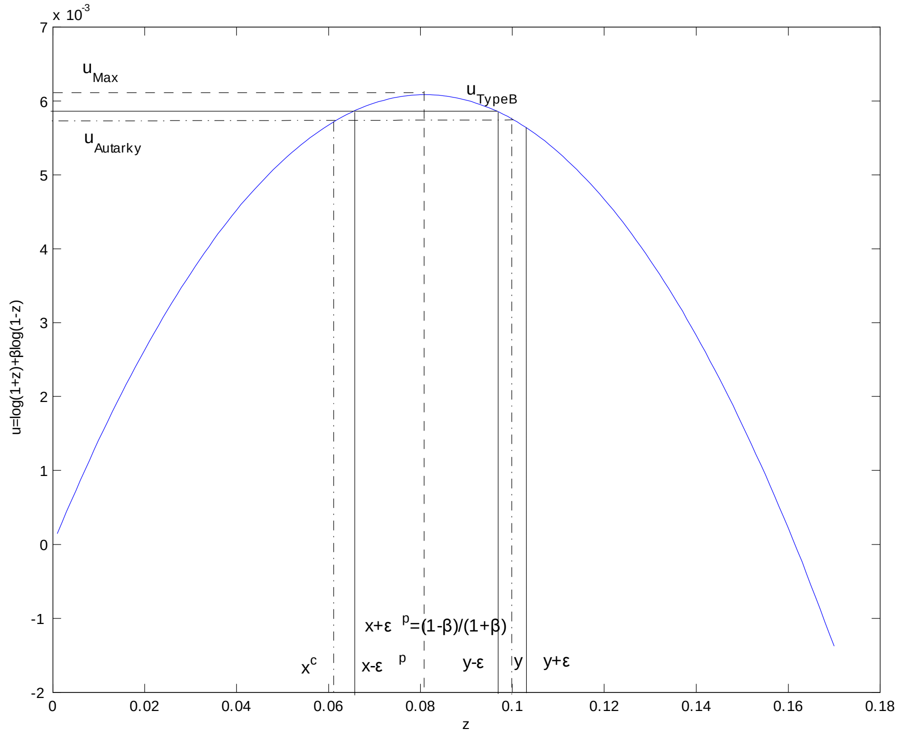

Figure 1: Trade Equilibrium Allocation

This figure illustrates the optimal consumption deviations for all three setups in country 1 with high initial endowment at period 0. All types in Jeske's and a centralized setup achieve, respectively, the ex-post utility level u_{Max} and $u_{Autarky}$. In this paper's setup, type A's utility level is equal to u_{Max}, while type B's utility level is denoted as u_{TypeB} The aggregate component of income fluctuation is characterized by $y = 0.1$. The idiosyncratic component of income fluctuation is characterized by $\varepsilon = 0.003$. The discount factor $\beta = 0.85$ satisfies both restrictions (15) and (16) given the value of y. Some international risk sharing can be supported.

Measure	Type n	Country m	
		$m = 1$	$m = 2$
$\frac{1}{2}$	$n = A$	$1 + y + \varepsilon$	$1 - y - \varepsilon$
$\frac{1}{2}$	$n = B$	$1 + y - \varepsilon$	$1 - y + \varepsilon$
$\frac{1}{2}(A + B)$		$1 + y$	$1 - y$

Table 1: Endowment Structure at Even Numbered Periods

Measure	Type n	Country m	
		$m = 1$	$m = 2$
$\frac{1}{2}$	$n = A$	$1 - y - \varepsilon$	$1 + y + \varepsilon$
$\frac{1}{2}$	$n = B$	$1 - y + \varepsilon$	$1 + y - \varepsilon$
$\frac{1}{2}(A + B)$		$1 + y$	$1 - y$

Table 2: Endowment Structure at Odd Numbered Periods