Forces and torques on spherical particles in a time-dependent electric field

R. Fuchs and F. Claro

(Sept. 17, 2008)

I. INTRODUCTION

The system has N spheres of radii a_i, $i = 1, \cdots, N$, composed of a material with a complex isotropic dielectric function $\varepsilon(\omega)$, in vacuum. If the applied electric field is associated with light, it is assumed that the size of the system is much smaller than the wavelength of light. This allows us (1) to use a nonretarded theory, in which the magnetic field is neglected; (2) to express the electric field as the gradient of a potential; (3) to approximate the applied electric field as a uniform field, which does not exert any force on the (uncharged) spheres. The electric potential is expressed by spherical harmonic expansions, with positive powers of r, using spherical coordinates with origins at the centers of the spheres, and the induced charges are represented by multipole moments q_{lmi} with arbitrarily high order l. In reference (1), we developed the theory used to find the q_{lmi}. We do not repeat the details of that theory here, but present only some of the equations and definitions needed for finding the forces and torques. Our present work is a generalization of reference (2), where the forces between two spheres were found using the dipole approximation, $l = 1$. Reference (3), a book by Gradshteyn and Ryzhik, has been used to find relations between associated Legendre functions that are needed in some derivations. The section and formula numbers in reference (3) are indicated by [GR 8.73xx x.]. Minor changes of the notation and indices have been made in the formulas.

II. MULTIPOLE MOMENTS AND COEFFICIENTS OF THE POTENTIAL

Assume that the external or applied electric field, as well as the induced charges and electric potential due to these charges, have an $\exp(-i\omega t)$ time dependence. If a complex quantity $f(\mathbf{r},t)$ is of the form $f(\mathbf{r},t) = f(\mathbf{r})\exp(-i\omega t)$, then $\mathrm{Re}\, f(\mathbf{r},t)$ is the physical value of $f(\mathbf{r},t)$. The total potential $V(\mathbf{r})$ can be written as a spherical harmonic expansion about the center of each sphere. In particular, the expansion for sphere i is

$$V_i(\mathbf{r}) = \sum_{l\,m} V_{lmi}\, r^l Y_{lm}(\theta,\phi), \tag{1}$$

where the origin of the polar coordinates (r,θ,ϕ) is at the center of sphere i. Note that $V_i(\mathbf{r})$ does not include the potential due to the charge induced on sphere i. The coefficient V_{lmi} is the sum of the contributions from the external potential and multipoles $q_{l'm'j}$ on the other spheres $j \neq i$:

$$V_{lmi} = V_{lmi}^{ext} + \sum_{l'm'j} B_{lmi}^{l'm'j} q_{l'm'j}, \tag{2}$$

where

$$B_{lmi}^{l'm'j} = (-1)^{l'+m'} \frac{Y^*_{l+l',m-m'}(\theta_{ij},\phi_{ij})}{R_{ij}^{l+l'+1}}(1-\delta_{ij})$$
$$\times\left[\frac{(4\pi)^3(l+l'+m-m')!\,(l+l'-m+m')!}{(2l+1)(2l'+1)(2l+2l'+1)(l+m)!(l-m)!(l'+m')!(l'-m')!}\right]^{1/2}. \tag{3}$$

Here $R_{ij}=\left|\mathbf{R}_j-\mathbf{R}_i\right|$ is the distance between the centers of sphere j at $\mathbf{R}_j$ and sphere i at $\mathbf{R}_i$, and θ_{ij}, ϕ_{ij} are the polar and azimuthal angles of the vector $\mathbf{R}_{ij}\equiv\mathbf{R}_j-\mathbf{R}_i$. The multipole coefficient q_{lmi} for sphere i defined by

$$q_{lmi} = \int \rho_i(\mathbf{r})\, r^l\, Y^*_{lm}(\theta,\phi)\, d^3r, \tag{4}$$

where $\rho_i(\mathbf{r})$, which is written as if it were a volume charge density, is actually the surface charge density induced on sphere i. The coefficients V_{lmi} and q_{lmi} are related by

$$q_{lmi} = -\frac{2l+1}{4\pi}\alpha_{li}V_{lmi}. \tag{5}$$

Here the multipole polarizability of sphere i is

$$\alpha_{li} = n_l^0 \frac{a_i^{2l+1}}{\left[\varepsilon(\omega)-1\right]^{-1}+n_l^0}, \tag{6}$$

where $n_l^0 = l/(2l+1)$. Eq. (6) is equivalent to Eq. (7) in reference (1). Note that in Eqs. (1), (2), (4), (5), (6), the quantities $V_i(\mathbf{r})$, V_{lmi}, q_{lmi}, $\rho_i(\mathbf{r})$, and α_{li} depend on the frequency ω because of the frequency dependence of the dielectric function $\varepsilon(\omega)$, but this dependence is not indicated explicitly. This is also true for the time-averaged forces and torques that are found below. As pointed out in references (1) and (2), it may be preferable to use the spectral variable $u=-\left[\varepsilon(\omega)-1\right]^{-1}$ instead of ω in Eq. (6), which avoids having to use $\varepsilon(\omega)$ for a particular material.

We shall assume that the external potential corresponds to a uniform electric field in the z direction, $\mathbf{E}^{ext}(\mathbf{r}) = E_z^{ext}\mathbf{k}$. Then for every sphere i, the external potential is $V_i^{ext}(\mathbf{r}) = -E_z^{ext}z$. Since $Y_{10}(\theta,\phi)=\sqrt{3/4\pi}\cos\theta$, Eq. (1) can be satisfied for the external potential only if $l=1$, $m=0$, so that $V_i^{ext}(\mathbf{r}) = -E_z^{ext}z = V_{10i}^{ext}\, r\sqrt{3/4\pi}\cos\theta = V_{10i}^{ext}\, z\sqrt{3/4\pi}$, or $V_{10i}^{ext} = -\sqrt{4\pi/3}\,E_z^{ext}$. The q_{lmi} are calculated using Eq. (16) in reference (1), restricting the sum to $l'=1$, $m'=0$, but summing over spheres (j).

III. FORCES ON SPHERES

The time-averaged force on sphere i is

$$\langle \mathbf{F}_i \rangle = \frac{1}{2}\mathrm{Re}\int \rho_i^*(\mathbf{r})\mathbf{E}(\mathbf{r})d^3r, \tag{7}$$

where $\mathbf{E}(\mathbf{r}) = -\nabla V_i(\mathbf{r})$. In order to find the x, y, and z components of the force, one must first find the components of $\mathbf{E}(\mathbf{r})$:

$$E_x = E_r \sin\theta \cos\phi + E_\theta \cos\theta \cos\phi - E_\phi \sin\phi, \tag{8}$$

$$E_y = E_r \sin\theta \sin\phi + E_\theta \cos\theta \sin\phi + E_\phi \cos\phi, \tag{9}$$

$$E_z = E_r \cos\theta - E_\theta \sin\theta. \tag{10}$$

In order to calculate the electric field components in Eqs. (8)-(10) from Eq. (1), it is useful to write the spherical harmonics in Eq. (1) in terms of Legendre functions using the well-known relation

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\,\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta). \tag{11}$$

Then Eq. (1) can be written as

$$V_i(\mathbf{r}) = \sum_{l\,m} A_{lmi} r^l P_l^m(\theta)\, e^{im\phi}, \tag{12}$$

where $A_{lmi} = V_{lmi}\sqrt{\frac{2l+1}{4\pi}\,\frac{(l-m)!}{(l+m)!}}$. Then the spherical components of the electric field are

$$E_r = -\frac{\partial V_i(\mathbf{r})}{\partial r} = -\sum_{l\,m} A_{lm\,i}\, l r^{l-1} P_l^m(\cos\theta) e^{im\phi}, \tag{13}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V_i(\mathbf{r})}{\partial\theta} = -\sum_{l\,m} A_{l\,m\,i}\; r^{l-1}\frac{\partial}{\partial\theta} P_l^m(\cos\theta) e^{im\phi}$$

$$= \sum_{l\,m} A_{lm\,i}\; r^{l-1}\sqrt{1-u^2}\,\frac{\partial}{\partial u} P_l^m(u) e^{im\phi}, \tag{14}$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V_i(\mathbf{r})}{\partial\phi} = -\sum_{l\,m} A_{lm\,i}\; r^{l-1}\frac{1}{\sqrt{1-u^2}} P_l^m(u)\, im e^{im\phi}, \tag{15}$$

where $\cos\theta = u$, $\sin\theta = \sqrt{1-u^2}$. It is useful calculate linear combinations of E_x and E_y:

$$E_+ = E_x + iE_y = \left(E_r \sin\theta + E_\theta \cos\theta + i\,E_\phi\right) e^{i\phi}, \tag{16}$$

$$E_- = E_x - iE_y = \left(E_r \sin\theta + E_\theta \cos\theta - i\,E_\phi\right) e^{-i\phi}. \tag{17}$$

Eq. (16), together with Eqs. (13) and (14), give

$$E_+ = -\sum_{l\,m} A_{l\,mi} r^{l-1}\left[l\sqrt{1-u^2}\, P_l^m(u) - u\sqrt{1-u^2}\,\frac{\partial}{\partial u} P_l^m(u) - \frac{m}{\sqrt{1-u^2}} P_l^m(u)\right] e^{i(m+1)\phi}. \tag{18}$$

Using the functional relations $\left(1-u^2\right)\frac{\partial}{\partial u} P_l^m(u) = (l+m) P_{l-1}^m(u) - l\,u\,P_l^m(u)$ [GR 8.733 1.2] and $(l-m)P_l^m(u) - u(l+m)P_{l-1}^m(u) = \sqrt{1-u^2}\; P_{l-1}^{m+1}(u)$ [GR 8.735 1.] and doing some algebra, it can be shown that the expression in square brackets in Eq. (18) is equal to $P_{l-1}^{m+1}(u)$. Therefore,

$$E_+ = -\sum_{l\,m} A_{lm\,i} r^{l-1} P_{l-1}^{m+1}(u) e^{i(m+1)\phi}. \tag{19}$$

If we use the expression for A_{lmi} and Eq. (11) , we obtain

$$E_+ = -\sum_{lm} V_{lmi}\, r^{l-1} \sqrt{\frac{2l+1}{2l-1}(l-m)(l-m-1)}\; Y_{l-1,m+1}(\theta,\phi). \tag{20}$$

Eq. (17), together with Eqs. (13) and (14), give

$$E_- = -\sum_{lm} A_{lmi} r^{l-1}\left[l\sqrt{1-u^2}\;P_l^m(u) - u\sqrt{1-u^2}\;\frac{\partial}{\partial u}P_l^m(u) + \frac{m}{\sqrt{1-u^2}}\;P_l^m(u)\right] e^{i(m+1)\phi}. \tag{21}$$

The functional relation for $\partial P_l^m(u)/\partial u$ given above [GR 8.733 1.2] and the recurrence relation $uP_{l-1}^m(u) - P_l^m(u) = (l+m-1)\sqrt{1-u^2}\;P_{l-1}^{m-1}(u)$ [GR 8.735 4.] can be used to show that the expression in square brackets in Eq. (21) is equal to $-(l+m)(l+m-1)P_{l-1}^{m-1}(u)$. We find

$$\begin{aligned} E_- &= \sum_{lm} A_{lmi}\, r^{l-1}(l+m)(l+m-1)\; P_{l-1}^{m-1}(u) e^{i(m-1)\phi} \\ &= \sum_{lm} V_{lmi}\, r^{l-1} \sqrt{\frac{2l+1}{2l-1}(l+m)(l+m-1)}\; Y_{l-1,\,m-1}(\theta,\phi). \end{aligned} \tag{22}$$

Finally, use Eq. (10) for E_z together with Eqs. (13) and (14), and the equation $\left(1-u^2\right)\frac{\partial}{\partial u}P_l^m(u) = (l+m)P_{l-1}^m(u) - l\,u\,P_l^m(u)$ [GR 8.733 1.2], to give

$$\begin{aligned} E_z &= -\sum_{lm} A_{lmi}\, r^{l-1}\left[luP_l^m(u) + \left(1-u^2\right)\frac{\partial}{\partial u}P_l^m(u)\right] e^{im\phi} \\ &= -\sum_{lm} A_{lmi}\, r^{l-1}\,(l+m)P_{l-1}^m(u)\, e^{im\phi} \\ &= -\sum_{lm} V_{lmi} r^{l-1} \sqrt{\frac{2l+1}{2l-1}(l-m)(l+m)}\; Y_{l-1,\,m}(\theta,\phi). \end{aligned} \tag{23}$$

The expressions for E_x and E_y are found by inverting the definitions of E_+ and E_- in Eqs. (16) and (17): $E_x = \frac{1}{2}\left(E_+ + E_-\right)$, $E_y = \frac{i}{2}\left(-E_+ + E_-\right)$. To find the expressions for the forces, it is useful to take the complex conjugate of Eq. (4) and replace $l \to l-1$:

$$q_{l-1,m,i} = \int \rho_i^*(\mathbf{r})\, r^{l-1}\, Y_{l-1,m}(\theta,\phi)\, d^3r. \tag{24}$$

We find the x, y, and z components of the force on sphere i using Eq. (7). The x component is

$$\begin{aligned} \left\langle F_{ix}\right\rangle &= \frac{1}{2}\mathrm{Re}\int \rho_i^*(\mathbf{r})\, E_x\, d^3r \\ &= \frac{1}{2}\mathrm{Re}\int \rho_i^*(\mathbf{r})\,\frac{1}{2}\left(E_+ + E_-\right) d^3r \\ &= -\,\frac{1}{4}\mathrm{Re}\int \rho_i^*(\mathbf{r}) \sum_{lm} V_{lmi}\, r^{l-1}\sqrt{\frac{2l+1}{2l-1}} \\ &\qquad \times\left[\sqrt{(l-m)(l-m-1)}\,Y_{l-1,\,m+1}(\theta,\phi) - \sqrt{(l+m)(l+m-1)}\,Y_{l-1,\,m-1}(\theta,\phi)\right] d^3r \end{aligned} \tag{25}$$

From the definition of the multipole moments, Eq. (24), one can write Eq. (25) as

$$\left\langle F_{ix}\right\rangle = -\frac{1}{4}\text{Re}\sum_{lm} V_{lmi}\sqrt{\frac{2l+1}{2l-1}}\left[\sqrt{(l-m)(l-m-1)}\; q^*_{l-1,m+1,i} - \sqrt{(l+m)(l+m-1)}\, q^*_{l-1,m-1,i}\right]. \tag{26}$$

There are two ways to find the force in terms of the multipole moments. The first way is to eliminate V_{lmi} as a function of q_{lmi} using Eq. (5). This gives

$$\begin{aligned}\left\langle F_{ix}\right\rangle &= -\frac{1}{4}\text{Re}\sum_{lm}\left[-\frac{4\pi}{2l+1}\frac{q_{lmi}}{\alpha_{li}}\right]\sqrt{\frac{2l+1}{2l-1}}\\ &\quad\times\left[\sqrt{(l-m)(l-m-1)}\; q^*_{l-1,m+1,i} - \sqrt{(l+m)(l+m-1)}\, q^*_{l-1,m-1,i}\right]\\ &= \pi\,\text{Re}\sum_{lm}(\alpha_{li})^{-1}\left[\sqrt{\frac{(l-m)(l-m-1)}{(2l+1)(2l-1)}}\,q_{lmi}\,q^*_{l-1,m+1,i} - \sqrt{\frac{(l+m)(l+m-1)}{(2l+1)(2l-1)}}\,q_{lmi}\,q^*_{l-1,m-1,i}\right]\end{aligned} \tag{27}$$

The second way to eliminate V_{lmi} is to use Eq. (2), omitting the term V^{ext}_{lmi} because a uniform external field does not exert a force on the sphere:

$$\begin{aligned}\left\langle F_{ix}\right\rangle &= -\frac{1}{4}\text{Re}\sum_{lm}\left[\sum_{l'm'j} B^{l'm'j}_{lmi} q_{l'm'j}\right]\sqrt{\frac{2l+1}{2l-1}}\\ &\quad\times\left[\sqrt{(l-m)(l-m-1)}\; q^*_{l-1,m+1,i} - \sqrt{(l+m)(l+m-1)}\, q^*_{l-1,m-1,i}\right]\\ &= -\frac{1}{4}\text{Re}\sum_{lm\,l'm'j} B^{l'm'j}_{lmi}\sqrt{\frac{2l+1}{2l-1}}\Big[\sqrt{(l-m)(l-m-1)}\, q_{l'm'j}\, q^*_{l-1,m+1,i}\\ &\qquad - \sqrt{(l+m)(l+m-1)}\, q_{l'm'j}\, q^*_{l-1,m-1,i}\Big]\end{aligned} \tag{28}$$

$$\begin{aligned}&= -\frac{1}{4}\text{Re}\sum_{lm\,l'm'j}\sqrt{\frac{2l+3}{2l+1}}\Big[B^{l'm'j}_{l+1,m-1,i}\sqrt{(l-m+2)(l-m+1)}\, q_{l'm'j}\, q^*_{lmi}\\ &\qquad - B^{l'm'j}_{l+1,m+1,i}\sqrt{(l+m+2)(l+m+1)}\, q_{l'm'j}\, q^*_{lmi}\Big]\end{aligned} \tag{29}$$

To derive Eq. (29) from Eq. (28), we have replaced $l \to l+1$ everywhere, $m \to m-1$ in the first term in square brackets, and $m \to m+1$ in the second term in square brackets. This changes the indices on the multipole moments to $l\,m\,i$ and $l'\,m'\,j$, without any $+1$ or -1 in the indices as in Eq. (28). The reason for making these replacements is to simplify the limits on the summation. There are no zero-order multipoles, so because of the multipole order $l-1$ in Eq. (28), the sum over l must be restricted to $l \geq 2$. Moreover, the magnitude of the second index must less than or equal to the first index. Therefore in the first term in the square brackets in Eq. (28), the sum over m is restricted to $-l \leq m \leq l-2$, and in the second term in the square brackets, $-(l-2) \leq m \leq l$. The other indices have the usual ranges $l' \geq 1$ and $-l' \leq m' \leq l$. In Eq. (29) both the primed and unprimed indices have the usual ranges $l' \geq 1$, $-l' \leq m' \leq l'$, $l \geq 1$, and $-l \leq m \leq l$. In Eq. (27) the ranges of the l and m indices are the same as in Eq. (28).

The y component of the time-averaged force on sphere i is

$$\left\langle F_{iy}\right\rangle = \frac{1}{2}\mathrm{Re}\int \rho_i^*(\mathbf{r})E_y\, d^3r$$
$$= \frac{1}{2}\mathrm{Re}\int \rho_i^*(\mathbf{r})\frac{i}{2}\left(-E_+ + E_-\right)d^3r$$
$$= \frac{1}{2}\mathrm{Re}\int \rho_i^*(\mathbf{r})\frac{i}{2}\sum_{lm}V_{lmi}r^{l-1}\sqrt{\frac{2l+1}{2l-1}}$$
$$\times\left[\sqrt{(l-m)(l-m-1)}Y_{l-1,\,m+1}(\theta,\phi)+\sqrt{(l+m)(l+m-1)}Y_{l-1,\,m-1}(\theta,\phi)\right]d^3r. \qquad (30)$$

Using Eq. (24) we get

$$\left\langle F_{iy}\right\rangle = \frac{1}{4}\mathrm{Re}\; i\sum_{lm}V_{lmi}\sqrt{\frac{2l+1}{2l-1}}\left[\sqrt{(l-m)(l-m-1)}\; q^*_{l-1,\,m+1}+\sqrt{(l+m)(l+m-1)}\, q^*_{l-1,\,m-1}\right]. \qquad (31)$$

The first way to eliminate V_{lmi} gives

$$\left\langle F_{iy}\right\rangle = \frac{1}{4}\mathrm{Re}\; i\sum_{lm}\left[-\frac{4\pi}{2l+1}\frac{q_{lmi}}{\alpha_{li}}\right]\sqrt{\frac{2l+1}{2l-1}}$$
$$\times\left[\sqrt{(l-m)(l-m-1)}\; q^*_{l-1,\,m+1,i}+\sqrt{(l+m)(l+m-1)}\, q^*_{l-1,m-1,i}\right]$$
$$= -\pi\,\mathrm{Re}\, i\sum_{lm}(\alpha_{li})^{-1}\left[\sqrt{\frac{(l-m)(l-m-1)}{(2l+1)(2l-1)}}q_{lmi}\, q^*_{l-1,\,m+1,i}+\sqrt{\frac{(l+m)(l+m-1)}{(2l+1)(2l-1)}}q_{lmi}\, q^*_{l-1,m-1,i}\right]. \qquad (32)$$

The second way to eliminate V_{lmi} gives

$$\left\langle F_{iy}\right\rangle = \frac{1}{4}\mathrm{Re}\; i\sum_{lm}\left[\sum_{l'm'j}B_{lmi}^{l'm'j}q_{l'm'j}\right]\sqrt{\frac{2l+1}{2l-1}}$$
$$\times\left[\sqrt{(l-m)(l-m-1)}\; q^*_{l-1,\,m+1,\,i}+\sqrt{(l+m)(l+m-1)}\, q^*_{l-1,m-1,\,i}\right]$$
$$= \frac{1}{4}\mathrm{Re}\; i\sum_{lm\,l'm'j}B_{lmi}^{l'm'j}\sqrt{\frac{2l+1}{2l-1}}\left[\sqrt{(l-m)(l-m-1)}\,q_{l'm'j}\,q^*_{l-1,\,m+1,i}\right.$$
$$\left.+\sqrt{(l+m)(l+m-1)}\,q_{l'm'j}\,q^*_{l-1,\,m-1,i}\right] \qquad (33)$$
$$= \frac{1}{4}\mathrm{Re}\, i\sum_{lm\,l'm'j}\sqrt{\frac{2l+3}{2l+1}}\left[B_{l+1,m-1,i}^{l'm'j}\sqrt{(l-m+2)(l-m+1)}\,q_{l'm'j}\,q^*_{lmi}\right.$$
$$\left.+B_{l+1,m+1,i}^{l'm'j}\sqrt{(l+m+2)(l+m+1)}\,q_{l'm'j}\,q^*_{lmi}\right]. \qquad (34)$$

The z component of the time-averaged force on sphere i is

$$\left\langle F_{iz}\right\rangle = \frac{1}{2}\mathrm{Re}\int \rho_i^*(\mathbf{r})E_z\, d^3r$$
$$= -\frac{1}{2}\mathrm{Re}\int \rho_i^*(\mathbf{r})\sum_{lm}V_{lmi}\,r^{l-1}\sqrt{\frac{2l+1}{2l-1}(l-m)(l+m)}\;Y_{l-1,m}(\theta,\phi)\, d^3r$$
$$= -\frac{1}{2}\mathrm{Re}\sum_{lm}V_{lmi}\sqrt{\frac{2l+1}{2l-1}(l-m)(l+m)}\;q^*_{l-1,m,\,i} \qquad (35)$$

The first way to eliminate V_{lmi} gives

$$\langle F_{iz}\rangle = -\frac{1}{2}\mathrm{Re}\sum_{lm}\left[-\frac{4\pi}{2l+1}\frac{q_{lmi}}{\alpha_{li}}\right]\sqrt{\frac{2l+1}{2l-1}(l-m)(l+m)}\; q^*_{l-1,m,i}$$

$$= 2\pi\ \mathrm{Re}\sum_{lm}(\alpha_{li})^{-1}\sqrt{\frac{(l-m)(l+m)}{(2l+1)(2l-1)}}\; q_{lmi}\, q^*_{l-1,m,i}\,. \qquad (36)$$

The second way to eliminate V_{lmi} gives

$$\langle F_{iz}\rangle = -\frac{1}{2}\mathrm{Re}\sum_{lm}\left[\sum_{l'm'j} B^{l'm'j}_{lmi}\, q_{l'm'j}\right]\sqrt{\frac{2l+1}{2l-1}(l-m)(l+m)}\; q^*_{l-1,m,i}$$

$$= -\frac{1}{2}\mathrm{Re}\sum_{lml'm'j} B^{l'm'j}_{lmi}\sqrt{\frac{2l+1}{2l-1}(l-m)(l+m)}\; q_{l'm'j}\, q^*_{l-1,m,i} \qquad (37)$$

$$= -\frac{1}{2}\mathrm{Re}\sum_{lml'm'j} B^{l'm'j}_{l+1,m,i}\sqrt{\frac{2l+3}{2l+1}(l+1-m)(l+1+m)}\; q_{l'm'j}\, q^*_{lmi}\,. \qquad (38)$$

Note that in the sum over m in Eqs. (36) and (37), m is restricted to the range $-(l-1) \le m \le (l-1)$, which is different from the range of m in Eqs. (28) and (33).

IV. TORQES ON SPHERES

The instantaneous torque on sphere i due to an electric field $\mathbf{E}$ acting on the charge $\rho_i(\mathbf{r})d^3r$ in the infinitesimal volume element d^3r is $d\tau = \mathbf{r}\times d\mathbf{F} = \mathbf{r}\times(\mathbf{E}\,\rho_i(\mathbf{r})d^3r)$, where $\mathbf{r}$ is the vector distance from the center of the sphere. The total instantaneous torque is $\tau_i = \int \rho_i(\mathbf{r})\,\mathbf{r}\times\mathbf{E}\,d^3r$. For harmonic time-dependent quantities, $\rho_i(\mathbf{r})$ and $\mathbf{E}$ are complex amplitudes, and the time-average torque is

$$\langle \vec{\tau}_i\rangle = \frac{1}{2}\mathrm{Re}\int \rho_i^*(\mathbf{r})\ \mathbf{r}\times\mathbf{E}\ d^3r, \qquad (39)$$

which has Cartesian components

$$\langle \tau_{ix}\rangle = \frac{1}{2}\mathrm{Re}\int \rho_i^*(\mathbf{r})\,(yE_z - zE_y)\,d^3r, \qquad (40)$$

$$\langle \tau_{iy}\rangle = \frac{1}{2}\mathrm{Re}\int \rho_i^*(\mathbf{r})\,(zE_x - xE_z)\,d^3r, \qquad (41)$$

$$\langle \tau_{iz}\rangle = \frac{1}{2}\mathrm{Re}\int \rho_i^*(\mathbf{r})\,(xE_y - yE_x)\,d^3r. \qquad (42)$$

We will use the field variables $E_+ = E_x + iE_y$ and $E_- = E_x - iE_y$ introduced in Eqs. (16) and (17), as well as the analogous distance variables $r_+ = x + iy$, $r_- = x - iy$. Inverting these relations gives $E_x = (1/2)[E_+ + E_-]$, $E_y = (1/2i)[E_+ - E_-]$, $x = (1/2)(r_+ + r_-)$, and $y = (1/2i)(r_+ - r_-)$. Then the bracketed expressions in Eqs. (40) and (41) can be written

$$\begin{aligned} y\,E_z - z\,E_y &= (1/2i)(r_+ - r_-)E_z - (z/2i)(E_+ - E_-) \\ &= (i/2)\left[(z\,E_+ - r_+E_z) - (z\,E_- - r_-E_z)\right] \\ &= (i/2)\left[W_+ - W_-\right] \end{aligned} \tag{43}$$

and

$$\begin{aligned} z\,E_x - x\,E_z &= (z/2)(E_+ - E_-) - (1/2)(r_+ - r_-)E_z \\ &= (1/2)\left[(z\,E_+ - r_+E_z) + (z\,E_- - r_-E_z)\right] \\ &= (1/2)\left[W_+ + W_-\right] \end{aligned} \tag{44}$$

where $W_+ = z\,E_+ - r_+E_z$, $W_- = zE_- - r_-E_z$. We also can write $r_+ = x + iy = r\sin\theta\,(\cos\phi + i\sin\phi)$ $= r\sin\theta\,e^{i\phi} = r\sqrt{1-u^2}\;e^{i\phi}$, $r_- = x - iy = r\sin\theta\,(\cos\phi - i\sin\phi) = r\sin\theta\,e^{-i\phi} = r\sqrt{1-u^2}\;e^{-i\phi}$, and $z = r\cos\theta = r\,u$. Using Eqs. (19) and (23) for E_+ and E_z, we have

$$z\,E_+ = -r\,u\sum_{l\,m} A_{l\,m\,i}\,r^{l-1}\,P_{l-1}^{m+1}(u)\,e^{\,i(m+1)\phi}, \tag{45}$$

$$r_+E_z = -r\sqrt{1-u^2}\;e^{i\phi}\sum_{l\,m} A_{l\,m\,i}\;r^{l-1}\,(l+m)\,P_{l-1}^{m}(u)\,e^{\,i\,m\,\phi}. \tag{46}$$

Eqs. (44), (45), and the relation $-u\,P_{l-1}^{m+1}(u) + (l+m)\sqrt{1-u^2}\,P_{l-1}^{m}(u) = -P_l^{m+1}(u)$ [GR 8.735 4.] give

$$W_+ = z\,E_+ - r_+E_z = \sum_{l\,m} A_{l\,m\,i}\,r^l\left[-u\,P_{l-1}^{m+1}(u) + (l+m)\sqrt{1-u^2}\,P_{l-1}^{m}(u)\right]e^{\,i(m+1)\phi} \tag{47}$$

$$= \sum_{l\,m} V_{l\,m\,i}\,\sqrt{\frac{2l+1}{4\pi}\,\frac{(l-m)!}{(l+m)!}}\;r^l\left[-P_l^{m+1}(u)\right]e^{\,i(m+1)\,\phi} \tag{48}$$

$$= -\sum_{l\,m} V_{l\,m\,i}\;r^l\sqrt{(l-m)(l+m+1)}\;Y_l^{m+1}(\theta,\phi). \tag{49}$$

Using Eqs. (22) and (23) for E_- and E_z, we have

$$z\,E_- = ru\sum_{l\,m} A_{l\,m\,i}\,r^{l-1}(l+m)\,(l+m-1)\;P_{l-1}^{m-1}(u)\,e^{\,i(m-1)\phi}, \tag{50}$$

$$r_-E_z = -r\sqrt{1-u^2}\;e^{-i\phi}\sum_{l\,m} A_{l\,m\,i}\;r^{l-1}(l+m)\,P_{l-1}^{m}(u)\,e^{\,i\,m\,\phi} \tag{51}$$

Eqs. (50), (51) and $u\,(l+m-1)\,P_{l-1}^{m-1}(u) + \sqrt{1-u^2}\;P_{l-1}^{m+1}(u) = (l-m+1)\,P_l^{m-1}(u)$ [GR 8.735 1.] give

$$W_- = zE_- - r_-E_z = \sum_{l\,m} A_{l\,m\,i}\,r^l(l+m)\left[u\,(l+m-1)\;P_{l-1}^{m-1}(u) + \sqrt{1-u^2}\;P_{l-1}^{m}(u)\right]e^{\,i(m-1)\phi} \tag{52}$$

$$= \sum_{l\,m} V_{l\,m\,i}\;\sqrt{\frac{2l+1}{4\pi}\,\frac{(l-m)!}{(l+m)!}}\;r^l\,(l+m)\left[(l-m+1)\,P_l^{m-1}(u)\right]e^{\,i(m-1)\phi} \tag{53}$$

$$= \sum_{l\,m} V_{l\,m\,i}\;r^l\sqrt{(l+m)(l-m+1)}\;Y_l^{m-1}(\theta,\phi). \tag{54}$$

To find $\langle\tau_{i\,x}\rangle$ and $\langle\tau_{i\,y}\rangle$, we must do the following two integrals:

$$\int \rho_i^*(\mathbf{r})\, W_+ d^3r = -\int \sum_{lm} V_{lmi}\ r^l \sqrt{(l-m)(l+m+1)}\, Y_l^{m+1}(\theta,\phi)\ \rho_i^*(\mathbf{r})\, d^3r$$
$$= -\sum_{lm} V_{lmi} \sqrt{(l-m)(l+m+1)}\ q^*_{l,\,m+1,\,i}\ ; \tag{55}$$

$$\int \rho_i^*(\mathbf{r})\, W_- d^3r = \int \sum_{lm} V_{lmi}\ r^l \sqrt{(l+m)(l-m+1)}\, Y_l^{m-1}(\theta,\phi)\rho_i^*(\mathbf{r})\, d^3r$$
$$= \sum_{lm} V_{lmi} \sqrt{(l+m)(l-m+1)}\ q^*_{l,\,m-1,\,i}\ ; \tag{56}$$

The torque $\langle \tau_{ix} \rangle$ is found by combining Eqs. (40), (43), (49), and (51):

$$\langle \tau_{ix} \rangle = \frac{1}{2}\mathrm{Re}\int \rho_i^*(\mathbf{r})\,(i/2)\left[W_+ - W_-\right] d^3r$$
$$= \frac{1}{4}\mathrm{Re}\sum_{lm} i V_{lmi}\ \left[-\sqrt{(l-m)(l+m+1)}\ q^*_{l,m+1,i} - \sqrt{(l+m)(l-m+1)}\ q^*_{l,m-1,i}\right]. \tag{57}$$

Eliminating V_{lmi} using Eq. (5), we find

$$\langle \tau_{ix} \rangle = \mathrm{Re}\sum_l \frac{\pi i}{(2l+1)\alpha_{li}} \left[\sum_{m=-l}^{l-1} \sqrt{(l-m)(l+m+1)}\ q_{lmi}\, q^*_{l,m+1,i} + \sum_{m=-l+1}^{l} \sqrt{(l+m)(l-m+1)}\ q_{lmi}\, q^*_{l,m-1,i}\right] \tag{58}$$

It is important to note the different limits of the two sums over m in Eq. (58); this difference is hidden in previous equations, in which the limits of the sums are not shown. Let the first sum in the square brackets in Eq. (58) be denoted by

$$S_{li} = \sum_{m=-l}^{l-1} \sqrt{(l-m)(l+m+1)}\ q_{lmi}\, q^*_{l,\,m+1,\,i}\ . \tag{59}$$

By replacing the index $m \to m+1$ in the second sum, one finds that the second sum in the square brackets is the complex conjugate of the first sum, so Eq. (58) can be written

$$\langle \tau_{ix} \rangle = \mathrm{Re}\sum_l \frac{\pi i}{(2l+1)\alpha_{li}} \left[S_{li} + S^*_{li}\right] \tag{60}$$

One can easily find the y component of the torque by comparing Eqs. (41) and (44) with Eqs. (40) and (43):

$$\langle \tau_{iy} \rangle = \frac{1}{2}\mathrm{Re}\int \rho_i^*(\mathbf{r})\,(1/2)\left[W_+ + W_-\right] d^3r \tag{61}$$
$$= \mathrm{Re}\sum_l \frac{\pi}{(2l+1)\alpha_{li}} \left[S_{li} - S^*_{li}\right] \tag{62}$$

The z component of the torque, Eq. (42), contains the expression

$$x\,E_y - y\,E_x = (1/2)(r_+ + r_-)\ (1/2i)(E_+ - E_-) - (1/2i)(r_+ - r_-)\ (1/2)(E_+ + E_-)$$
$$= (1/2i)\left[r_- E_+ - r_+ E_-\right]. \tag{63}$$

We have

$$r_- E_+ = -r\sqrt{1-u^2}\, e^{-i\phi}\sum_{l\,m} A_{l\,m\,i}\, r^{l-1} P_{l-1}^{m+1}(u) e^{i(m+1)\phi}$$

$$= -\sum_{l\,m} A_{l\,m\,i}\, r^{l}\sqrt{1-u^2}\, P_{l-1}^{m+1}(u) e^{i m\phi}, \tag{64}$$

$$r_+ E_- = r\sqrt{1-u^2}\, e^{i\phi}\sum_{l\,m} A_{l\,m\,i}\, r^{l-1}(l+m)(l+m-1)\, P_{l-1}^{m-1}(u) e^{i(m-1)\phi}$$

$$= \sum_{l\,m} A_{l\,m\,i}\, r^{l}(l+m)(l+m-1)\sqrt{1-u^2}\, P_{l-1}^{m-1}(u) e^{i m\phi}, \tag{65}$$

$$r_- E_+ - r_+ E_- = -\sum_{l\,m} A_{l\,m\,i}\, r^{l}\left[\sqrt{1-u^2}\, P_{l-1}^{m+1}(u) + (l+m)(l+m-1)\sqrt{1-u^2}\, P_{l-1}^{m-1}(u)\right] e^{i m\phi} \tag{66}$$

Using $(l+m-1)\sqrt{1-u^2}\, P_{l-1}^{m-1}(u) = u\, P_{l-1}^{m}(u) - P_{l}^{m}(u)$ [GR 8.735 4], the expression in square brackets in Eq. (66), which we denote by $C,$ becomes

$$C = \sqrt{1-u^2}\, P_{l-1}^{m+1}(u) + (l+m)\left(u P_{l-1}^{m}(u) - P_{l}^{m}(u)\right)$$

$$= (l+m)u\, P_{l-1}^{m}(u) + \sqrt{1-u^2}\, P_{l-1}^{m+1}(u) - (l+m) P_{l}^{m}(u). \tag{67}$$

Since $(l+m)\,u\, P_{l-1}^{m}(u) + \sqrt{1-u^2}\, P_{l-1}^{m+1}(u) = (l-m) P_{l}^{m}(u)$ [GR 8.375 1]. we get

$$C = (l-m)P_{l}^{m}(u) - (l+m)P_{l}^{m}(u)$$

$$= -2mP_{l}^{m}(u). \tag{68}$$

Finally, using Eqs. (11), (12), (66), and (68), we find

$$r_- E_+ - r_+ E_- = \sum_{lm} A_{l\,m\,i}\, r^{l}\, 2m\, P_{l}^{m}(u) e^{i m\phi} \tag{69}$$

$$= \sum_{lm} V_{l\,m\,i}\, r^{l}\, 2m\, Y_{lm}(\theta,\phi). \tag{70}$$

From Eqs. (42), (63), and (70), the z component of the torque is

$$\left\langle \tau_{i\,z}\right\rangle = \frac{1}{2}\mathrm{Re}\int \rho_i^*(\mathbf{r})\, \frac{1}{2\,i}\left[r_- E_+ - r_+ E_-\right] d^3 r$$

$$= \frac{1}{2}\mathrm{Re}\int \rho_i^*(\mathbf{r})\, \frac{1}{2i}\sum_{lm} V_{l\,m\,i}\, r^{l}\, 2m Y_{l\,m}(\theta,\phi) d^3 r. \tag{71}$$

We first use Eq. (24) (with $l-1$ replaced by l) and then use Eq. (5) to eliminate $V_{l\,m\,i}$. The result is

$$\left\langle \tau_{i\,z}\right\rangle = \frac{1}{2}\mathrm{Re}\, \frac{1}{i}\sum_{l} V_{l\,m\,i}\, m\, q_{l\,m\,i}^*$$

$$= \frac{1}{2}\mathrm{Re}\, \frac{1}{i}\sum_{lm}\left[-\frac{4\pi}{2l+1}\,\frac{q_{l\,m\,i}}{\alpha_{l\,i}}\right] m\, q_{l\,m\,i}^*$$

$$= \mathrm{Re}\sum_{lm} \frac{2\pi\, i}{2l+1}\,\frac{m}{\alpha_{l\,i}}\, q_{l\,m\,i}\, q_{l\,m\,i}^*. \tag{72}$$

IV. DISCUSSION

Two procedures, depending on the method for eliminating the potential coefficients V_{lmi}, were used to find the forces. The first procedure, which used Eq. (5), gave Eqs. (27), (32), and (36) for the components of the force on sphere i. The second procedure, which used Eq. (2) (omitting the term V_{lmi}^{ext}), gave Eqs. (29), (34), and (38). The first procedure gave expressions that involve sums over only two indices l, m; however one must be careful to choose the correct limits on the indices. The second procedure gave expressions that involve sums over five indices l, m, l', m', j, which may increase computation times. However the final expressions for the forces are much more intuitive, since the force on any sphere i is written as a sum of contributions from other spheres j.

The torques were found using only the first procedure to eliminate V_{lmi}. The second procedure could also be used. However, since a uniform electric field can exert a torque on a sphere, the external potential term $V_{lmi}^{ext} = -\sqrt{4\pi/3}\, E_z^{ext}\, \delta_{l1}\delta_{m0}$ in Eq. (2) must be included.

The final equations for the torque show that if there is no dissipation in the sphere material - that is, if $\varepsilon(\omega)$ and α_{li} are real, then the torque is zero. This is clear from Eq. (60), where $S_{li} + S_{li}^*$ is real and the factor i makes the entire sum imaginary, from Eq. (62), where $S_{li} - S_{li}^*$ is imaginary and no factor i is present, and from Eq. (72), where the factor i makes the entire sum imaginary.

REFERENCES

1 R. Fuchs and F. Claro, Phys. Rev. B 39, 3875 (1989)

2 R. Fuchs and F. Claro, Appl. Phys. Lett. 85, 3280 (2004)

3 L. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, New York 1965)