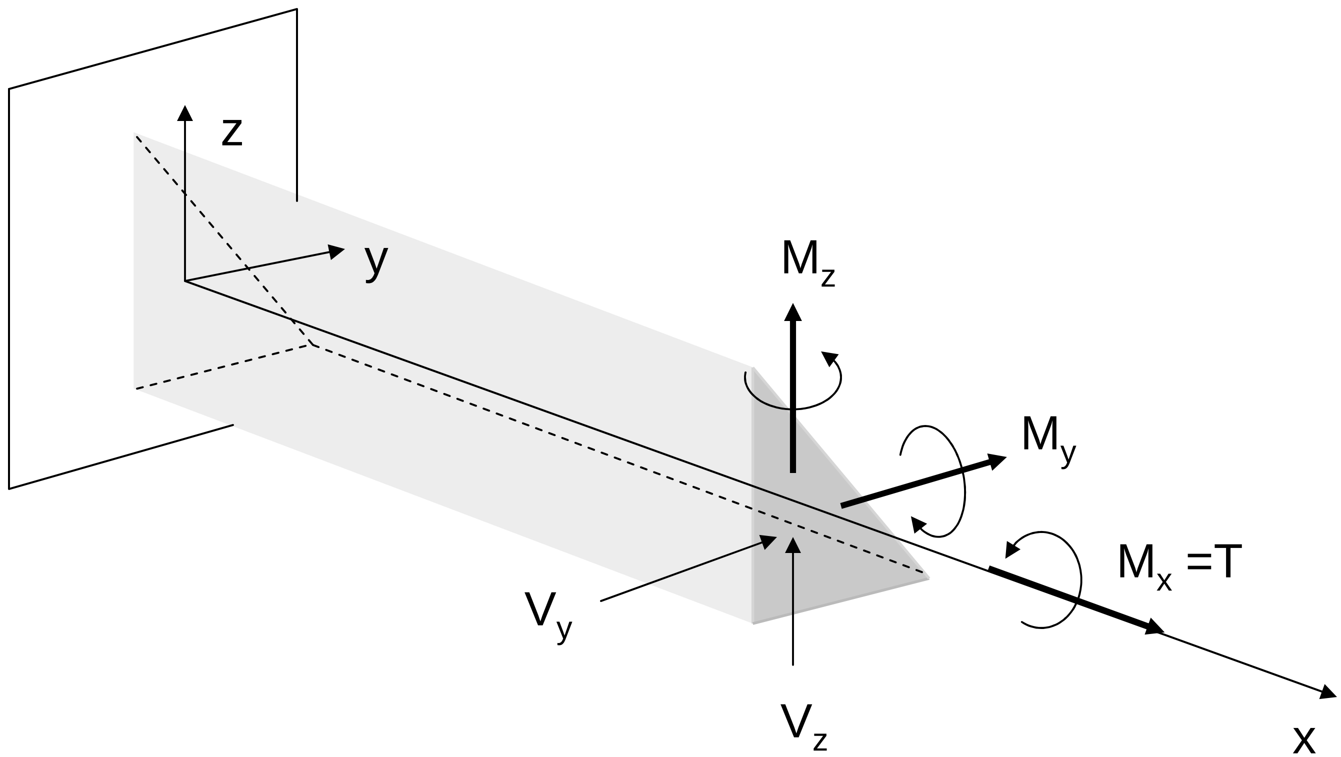
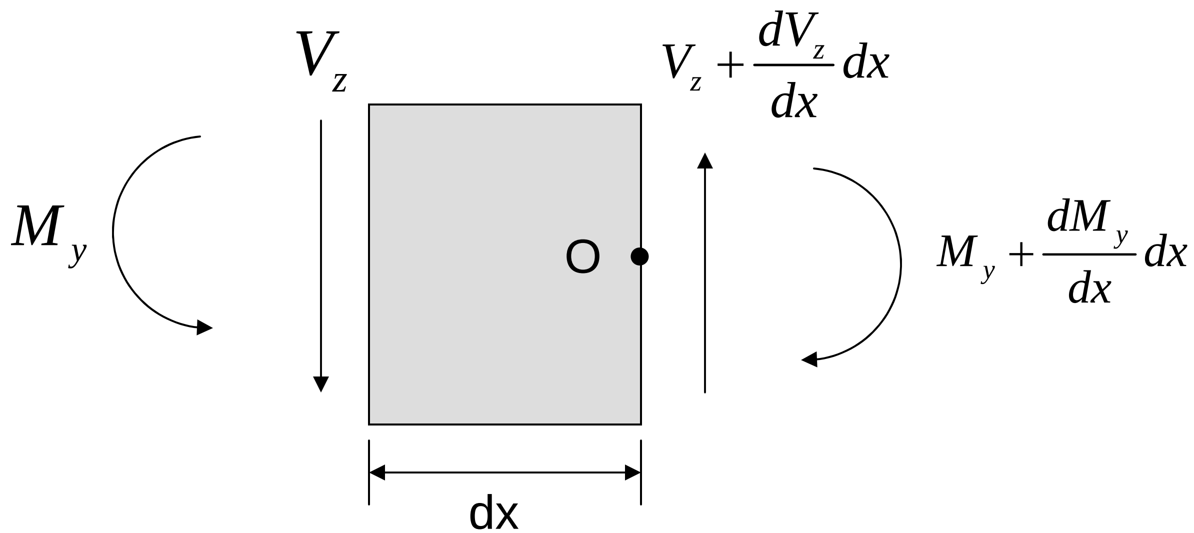
Combined bending of unsymmetrical beams

Internal forces and moments are defined positive as shown



Looking down the negative y -axis

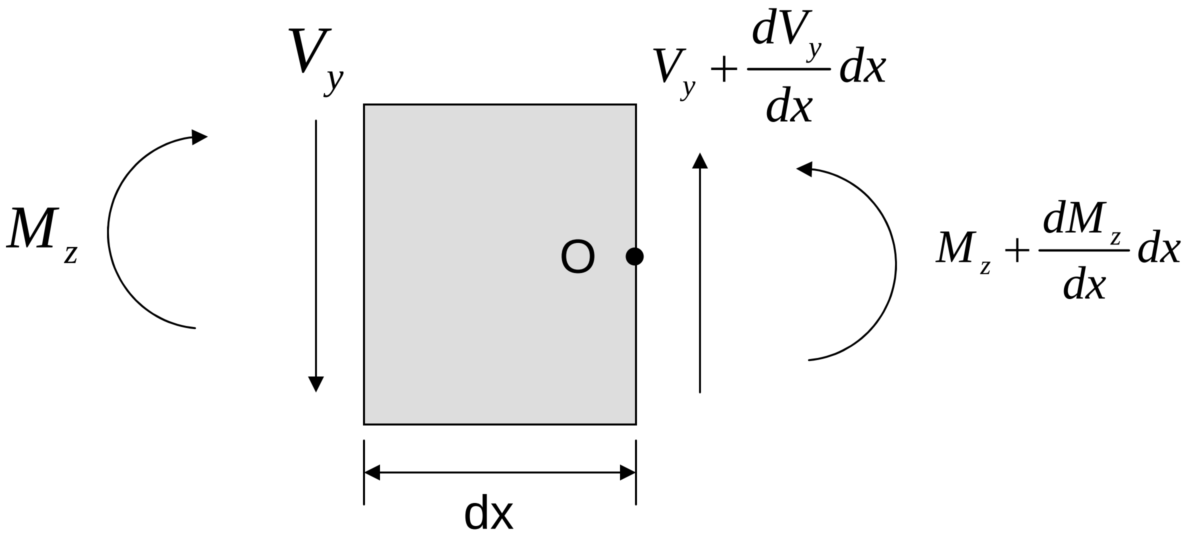


$+$ $\quad \sum M_O = 0$

$$M_y + V_z dx - \left(M_y + \frac{dM_y}{dx}dx \right) = 0$$

$$\Rightarrow \quad \boxed{\frac{dM_y}{dx} = V_z}$$

Looking down the z -axis



$$\circlearrowleft + \quad \sum M_O = 0$$

$$-M_z + V_y dx + \left(M_z + \frac{dM_z}{dx} dx \right) = 0$$

$$\Rightarrow \quad \boxed{\frac{dM_z}{dx} = -V_y}$$

Assume that plane sections remain plane but that is bending induced about both the y and z axes even if the moment might only be about one axis

$$u_x = -z\frac{dw}{dx} - y\frac{dv}{dx}$$

w is the displacement of the neutral axis in the z-direction

v is the displacement of the neutral axis in the y-direction

Thus, the axial strain developed is

$$e_{xx} = \frac{\partial u_x}{\partial x} = -z\frac{d^2 w}{dx^2} - y\frac{d^2 v}{dx^2}$$

curvatures of the neutral axis

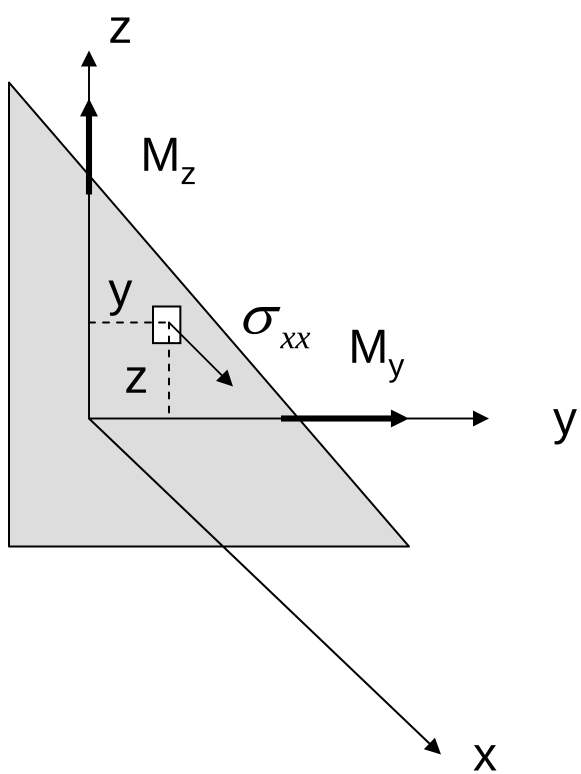
If σ_{xx} is the only major stress

$$\sigma_{xx} = Ee_{xx} = E\left(-z\frac{d^2 w}{dx^2} - y\frac{d^2 v}{dx^2}\right)$$

Note that there are also strains

$$e_{yy} = e_{zz} = -\nu e_{xx} = \nu\left(z\frac{d^2 w}{dx^2} + y\frac{d^2 v}{dx^2} \right)$$

Relationship between the flexural stress and the internal moments



$$\int \sigma_{xx} dA = 0 \qquad (1)$$

$$\int y\sigma_{xx} dA = -M_z \qquad (2)$$

$$\int z\sigma_{xx} dA = M_y \qquad (3)$$

From (1)

$$-E\frac{d^2w}{dx^2}\int zdA - E\frac{d^2v}{dx^2}\int ydA = 0$$

$$\Rightarrow \quad \int zdA = \int ydA = 0$$

i.e. neutral axis is at the centroid of the cross section

From (2)

$$E\frac{d^2w}{dx^2}\int yzdA + E\frac{d^2v}{dx^2}\int y^2 dA = M_z$$

$$\Rightarrow \quad E\frac{d^2w}{dx^2}I_{yz} + E\frac{d^2v}{dx^2}I_{zz} = M_z$$

From (3)

$$-E\frac{d^2w}{dx^2}\int z^2 dA - E\frac{d^2v}{dx^2}\int yzdA = M_y$$

$$\Rightarrow \quad -E\frac{d^2w}{dx^2}I_{yy} - E\frac{d^2v}{dx^2}I_{yz} = M_y$$

$$M_z = EI_{yz}\frac{d^2 w}{dx^2} + EI_{zz}\frac{d^2 v}{dx^2}$$

$$M_y = -EI_{yy}\frac{d^2 w}{dx^2} - EI_{yz}\frac{d^2 v}{dx^2}$$

solving these two equations for the curvatures

$$\frac{d^2 v}{dx^2} = \frac{\left(M_z I_{yy} + M_y I_{yz}\right)}{E\left(I_{yy} I_{zz} - I_{yz}^2\right)}$$

$$\frac{d^2 w}{dx^2} = \frac{-\left(M_y I_{zz} + M_z I_{yz}\right)}{E\left(I_{yy} I_{zz} - I_{yz}^2\right)}$$

and placing these curvature into the flexural stress relation gives

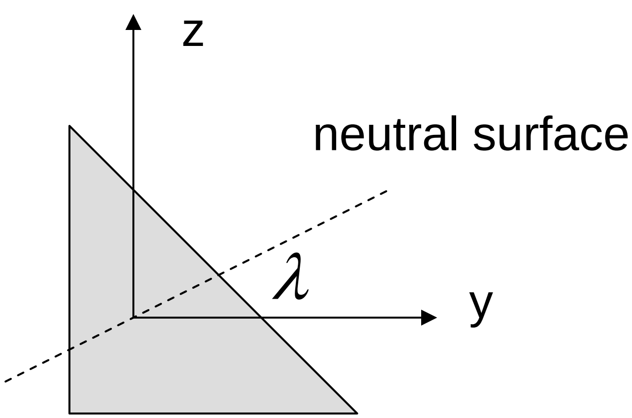
$$\sigma_{xx} = \frac{\left(M_y I_{zz} + M_z I_{yz}\right)z - \left(M_z I_{yy} + M_y I_{yz}\right)y}{\left(I_{yy} I_{zz} - I_{yz}^2\right)}$$

$$\sigma_{xx} = \frac{\left(M_y I_{zz} + M_z I_{yz}\right) z - \left(M_z I_{yy} + M_y I_{yz}\right) y}{\left(I_{yy} I_{zz} - I_{yz}^2\right)}$$

Note: If either the y or z axis is a plane of symmetry then I_{yz} =0 and we find

$$\sigma_{xx} = \frac{M_y z}{I_{yy}} - \frac{M_z y}{I_{zz}}$$

The neutral surface is located where σ_{xx} = 0 $\Rightarrow$ $z = \tan\lambda \; y$

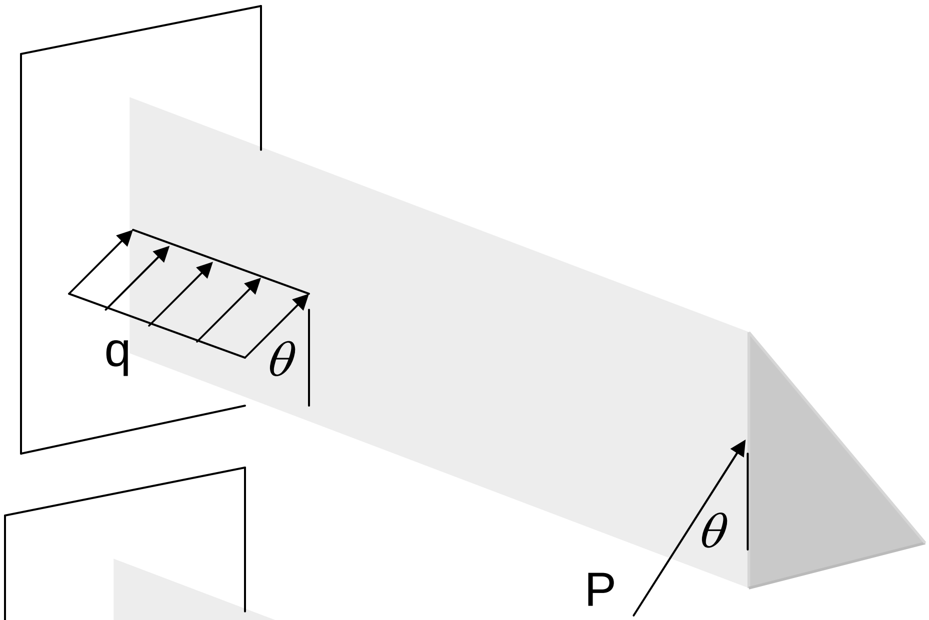


where

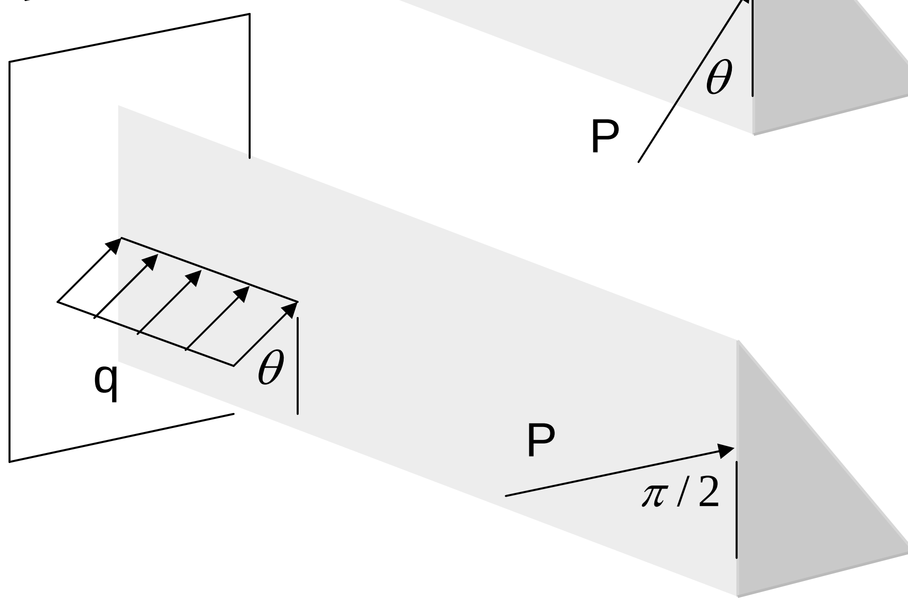
$$\tan\lambda = \frac{M_z I_{yy} + M_y I_{yz}}{M_y I_{zz} + M_z I_{yz}} = \frac{\left(M_z / M_y\right) I_{yy} + I_{yz}}{I_{zz} + \left(M_z / M_y\right) I_{yz}}$$

Although the bending moments may individually be functions of x, if the ratio M_z / M_y is independent of x (called single plane of loading), then the angle of the neutral axis will be a constant in the beam. Otherwise, the angle will vary with x.

single plane of loading



general loading



Flexure Stress Expressions

(1) $$\sigma_{xx} = \frac{\left(M_y I_{zz} + M_z I_{yz}\right) z - \left(M_z I_{yy} + M_y I_{yz}\right) y}{\left(I_{yy} I_{zz} - I_{yz}^2\right)}$$

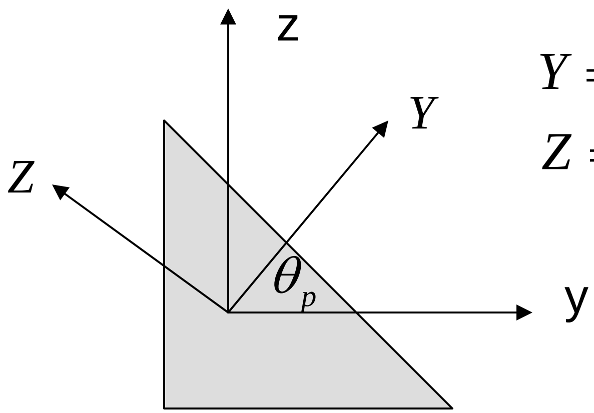
(2) can use $\tan\lambda = \dfrac{\left(M_z / M_y\right) I_{yy} + I_{yz}}{I_{zz} + \left(M_z / M_y\right) I_{yz}}$ to eliminate explicit dependency

on M_z in equation (1) and obtain

$$\sigma_{xx} = \frac{M_y \left(z - \tan\lambda\, y\right)}{I_{yy} - I_{yz} \tan\lambda}$$

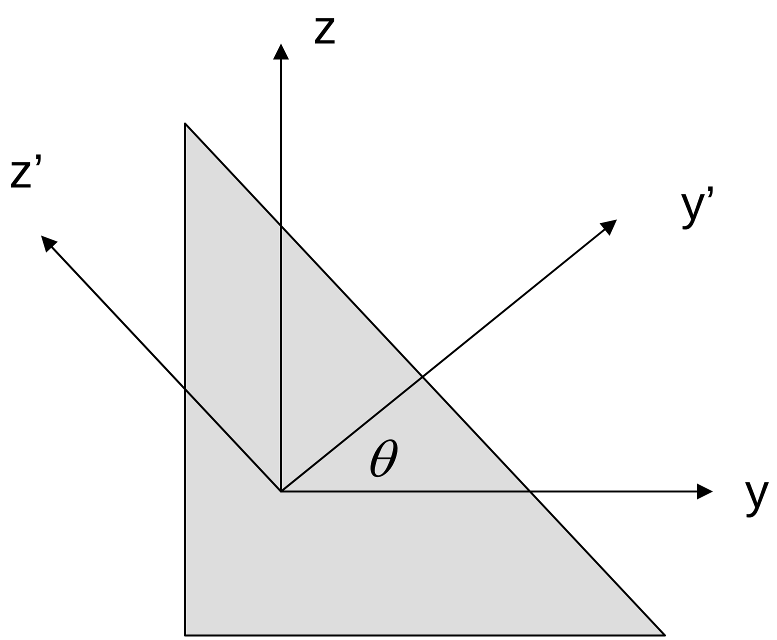
Note: M_z is still in here

(3) can use principal axes of the cross section



$$Y = z\sin\theta + y\cos\theta$$
$$Z = z\cos\theta - y\sin\theta$$

$$\boxed{\sigma_{xx} = \frac{M_Y Z}{I_{YY}} - \frac{M_Z Y}{I_{ZZ}}}$$



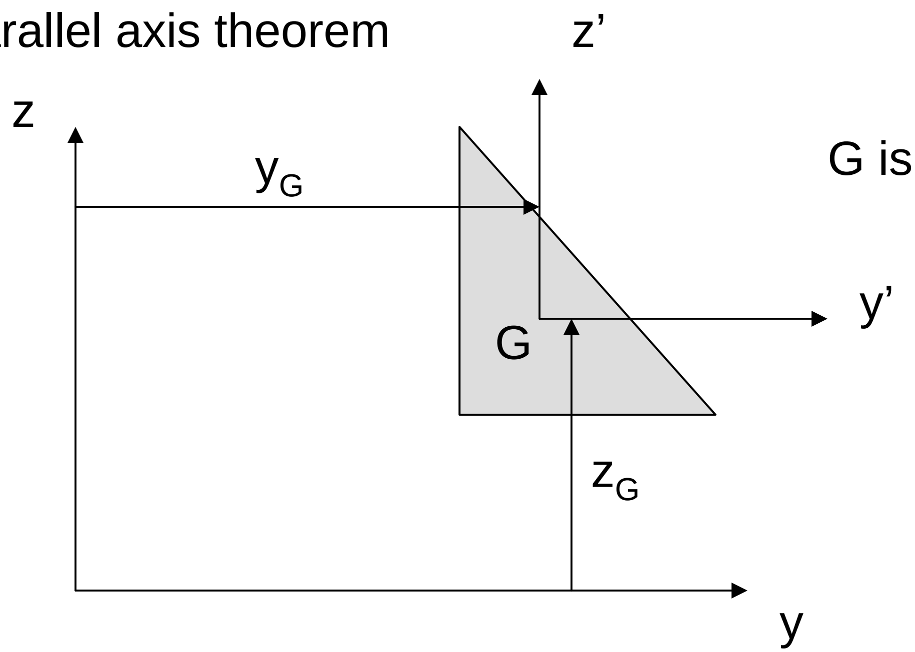
transformation relations for area moments

$$I_{y'y'} = \frac{I_{yy} + I_{zz}}{2} + \frac{I_{yy} - I_{zz}}{2}\cos(2\theta) - I_{yz}\sin(2\theta)$$

$$I_{z'z'} = \frac{I_{yy} + I_{zz}}{2} - \frac{I_{yy} - I_{zz}}{2}\cos(2\theta) + I_{yz}\sin(2\theta)$$

$$I_{y'z'} = \frac{I_{yy} - I_{zz}}{2}\sin(2\theta) + I_{yz}\cos(2\theta)$$

parallel axis theorem



G is the centroid of A

$$I_{zz} = \int y^2 dA = \int \left(y_G + y'\right)^2 dA = \left(I_{z'z'}\right)_G + A y_G^2$$

similarly

$$I_{yy} = \left(I_{y'y'}\right)_G + A z_G^2$$

$$I_{yz} = \left(I_{y'z'}\right)_G + A y_G z_G$$

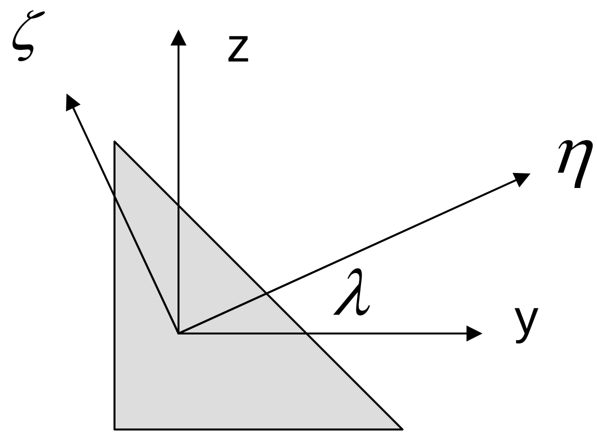
principal area moments and principal directions

$$\tan\left(2\theta_p\right)=\frac{-2I_{yz}}{I_{yy}-I_{zz}}$$

$$I_{p1,p2}=\frac{I_{yy}+I_{zz}}{2}\pm\sqrt{\left(\frac{I_{yy}-I_{zz}}{2}\right)^2+I_{yz}^2}$$

(4) can use neutral surface coordinates

(zeta)



(eta)

can write $$\sigma_{xx} = \frac{M_y I_{zz} + M_z I_{yz}}{I_{yy} I_{zz} - I_{yz}^2} \left(z - \tan\lambda \; y \right)$$

but $$z - \tan\lambda \; y = \frac{z\cos\lambda - y\sin\lambda}{\cos\lambda} = \frac{\zeta}{\cos\lambda}$$

so

$$\sigma_{xx} = \frac{M_y I_{zz} + M_z I_{yz}}{I_{yy} I_{zz} - I_{yz}^2} \frac{\zeta}{\cos\lambda}$$

displacements of the neutral axis

we must integrate

$$\frac{d^2v}{dx^2} = \frac{\left(M_z I_{yy} + M_y I_{yz}\right)}{E\left(I_{yy} I_{zz} - I_{yz}^2\right)}$$

$$\frac{d^2w}{dx^2} = \frac{-\left(M_y I_{zz} + M_z I_{yz}\right)}{E\left(I_{yy} I_{zz} - I_{yz}^2\right)}$$

twice to obtain the displacements, using boundary conditions on both v and w

$$\frac{\dfrac{d^2v}{dx^2}}{\dfrac{d^2w}{dx^2}} = \frac{-\left(M_z I_{yy} + M_y I_{yz}\right)}{\left(M_y I_{zz} + M_z I_{yz}\right)} = -\tan\lambda$$

$$\frac{d^2v}{dx^2} = -\tan\lambda \frac{d^2w}{dx^2}$$

If λ is a constant and we let $\frac{d^2w}{dx^2} = -F\left(x\right)$ we find, on integration

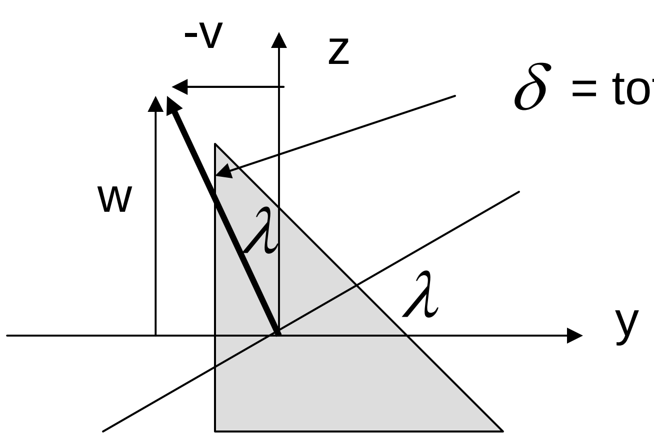
$$w = -\iint F(x)\,dxdx + C_1 x + C_2$$

$$\frac{-v}{\tan\lambda} = -\iint F(x)\,dxdx + C_3 x + C_4$$

If the boundary conditions on v, w are the same so that

$C_1 = C_3, C_2 = C_4$ then

$\frac{w}{v} = \frac{-1}{\tan\lambda}$ or $\tan\lambda = \frac{-v}{w}$



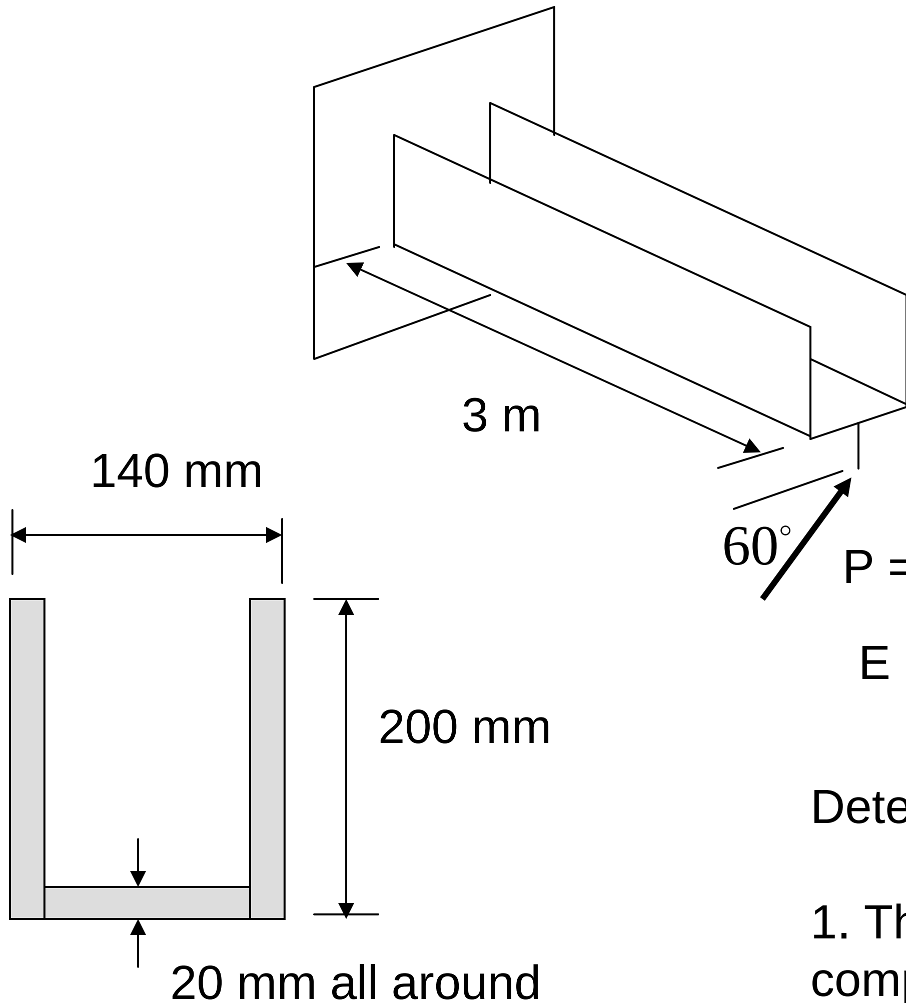
δ = total displacement

Thus, in this case the total displacement

$\delta = \sqrt{w^2 + u^2}$ is perpendicular to the

neutral surface.

Example 1



P = 12 kN (through the shear center)

E = 72 GPa

Determine:

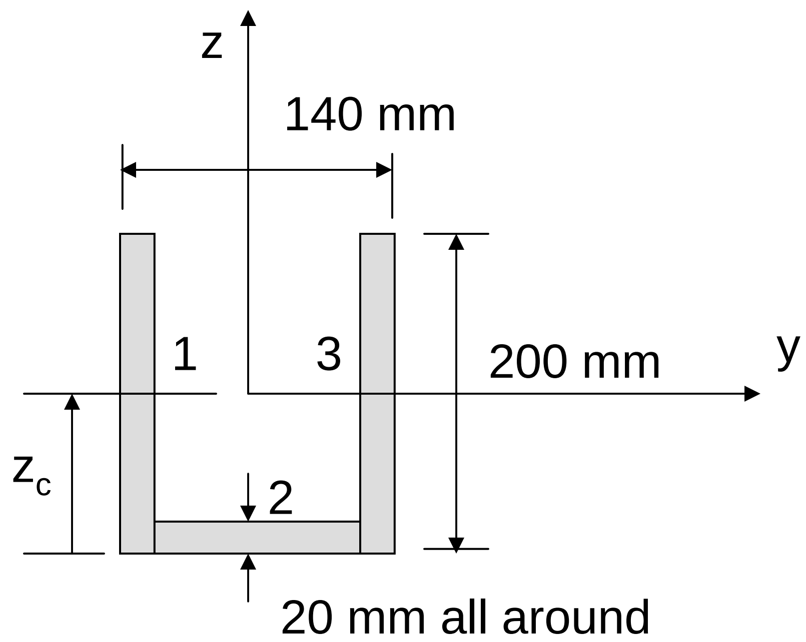
1. The maximum tensile and compressive stresses and their location

2. The total deflection at the load P

$$z_c = \frac{z_1 A_1 + z_2 A_2 + z_3 A_3}{A_1 + A_2 + A_3}$$

$$= \frac{2 \times (100)(4000) + (10)(2000)}{10000}$$

$$= 82\ mm$$



$$I_{yz} = 0$$

$$I_{zz} = 2 \times \left[\frac{1}{12}(200)(20)^3 + (200)(20)(70-10)^2 \right] + \left[\frac{1}{12}(20)(100)^3 \right]$$

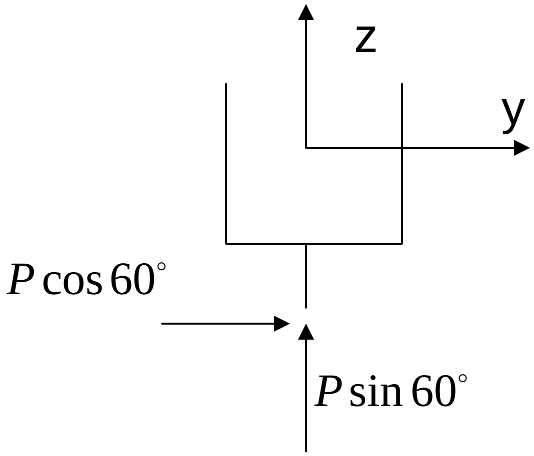
$$= 30.73 \times 10^6\ mm^4$$

$$I_{yy} = 2 \times \left[\frac{1}{12}(20)(200)^3 + (200)(20)(100-82)^2 \right]$$

$$+ \left[\frac{1}{12}(100)(20)^3 + (100)(20)(82-10)^2 \right]$$

$$= 39.69 \times 10^6\ mm^4$$

Maximum bending moments are at the wall



$$M_y = -300P\sin 60^\circ = -31.17\times 10^6\ N-mm$$

$$M_z = 3000P\cos 60^\circ = 18\times 10^6\ N-mm$$

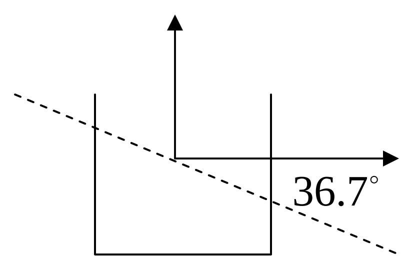
$$\frac{M_z}{M_y} = -\cot 60^\circ = -0.577$$

$$\tan\lambda = \frac{\left(M_z/M_y\right)I_{yy} + I_{yz}}{I_{zz} + \left(M_z/M_y\right)I_{yz}}$$

$$= \left(-0.577\right)\left(\frac{39.69}{30.73}\right) = -0.746$$

$\Rightarrow$ $\lambda = -0.640\ rad$

$= -36.7^\circ$

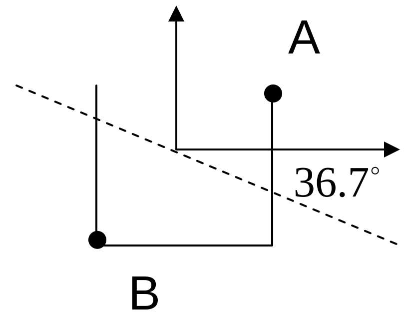


N-mm mm

$$\sigma_{xx} = \frac{M_y \left(z - \tan\lambda\, y\right)}{I_{yy} - \cancel{I_{yz}} \tan\lambda}$$ mm^4

$$= \frac{-31.17}{39.69}\left(z + 0.746\, y\right)$$

$$= -0.785\, z - 0.585\, y$$ N/mm^2 = MPa



At point A: $z = 200 - 82 = 118\ mm$

$y = 70\ mm$

$$\left(\sigma_{xx}\right)_A = -\left(0.785\right)\left(118\right) - \left(0.585\right)\left(70\right)$$
$$= -133.6\ MPa$$

At point B: $z = -82\ mm$

$y = -70\ mm$

$$\left(\sigma_{xx}\right)_B = -\left(0.785\right)\left(-82\right) - \left(0.585\right)\left(-70\right)$$
$$= 105.3\ MPa$$

alternately, with principal values

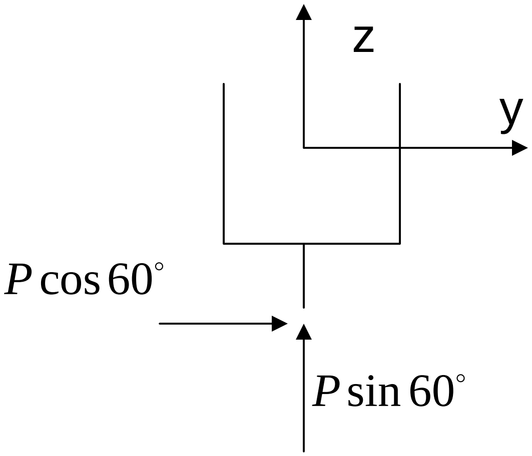
$$\sigma_{xx} = \frac{M_y z}{I_{yy}} - \frac{M_z y}{I_{zz}}$$

$$= -\frac{31.17}{39.69} z - \frac{18}{30.73} y$$

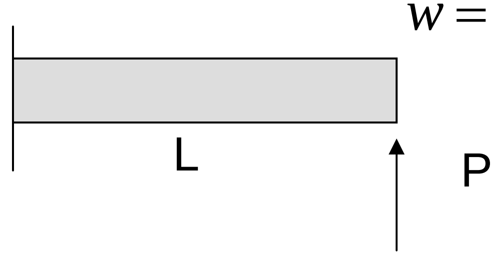
$$= -0.785z - 0.585y$$

which is same as before

Now, compute deflection at the load



$$\frac{d^2 w}{dx^2} = \frac{-M_y}{EI_{yy}}$$



$$w = \frac{PL^3}{3EI}$$

$$w = \frac{\left(P\sin 60^\circ\right)L^3}{3EI_{yy}}$$

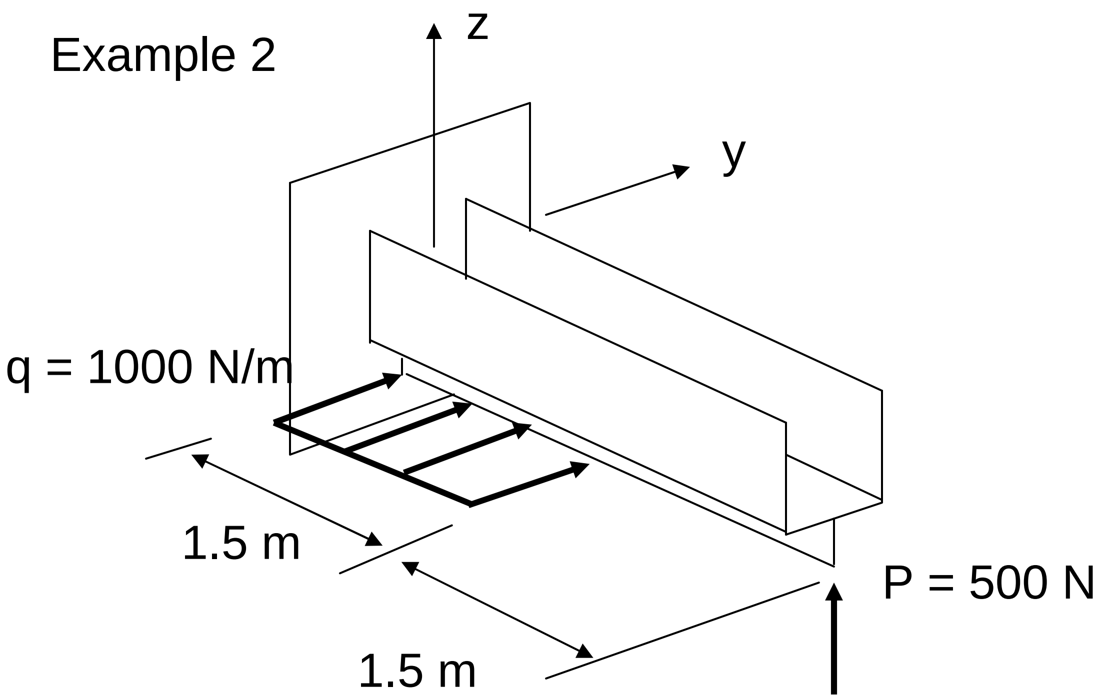
$$= \frac{\left(12{,}000\sin 60^\circ\right)\left(3000\right)^3}{3\left(72\times10^3\right)\left(39.69\times10^6\right)}$$

$$= 32.72\ mm$$

$$v = -\tan\lambda\ w = 0.745w = 24.38\ mm$$

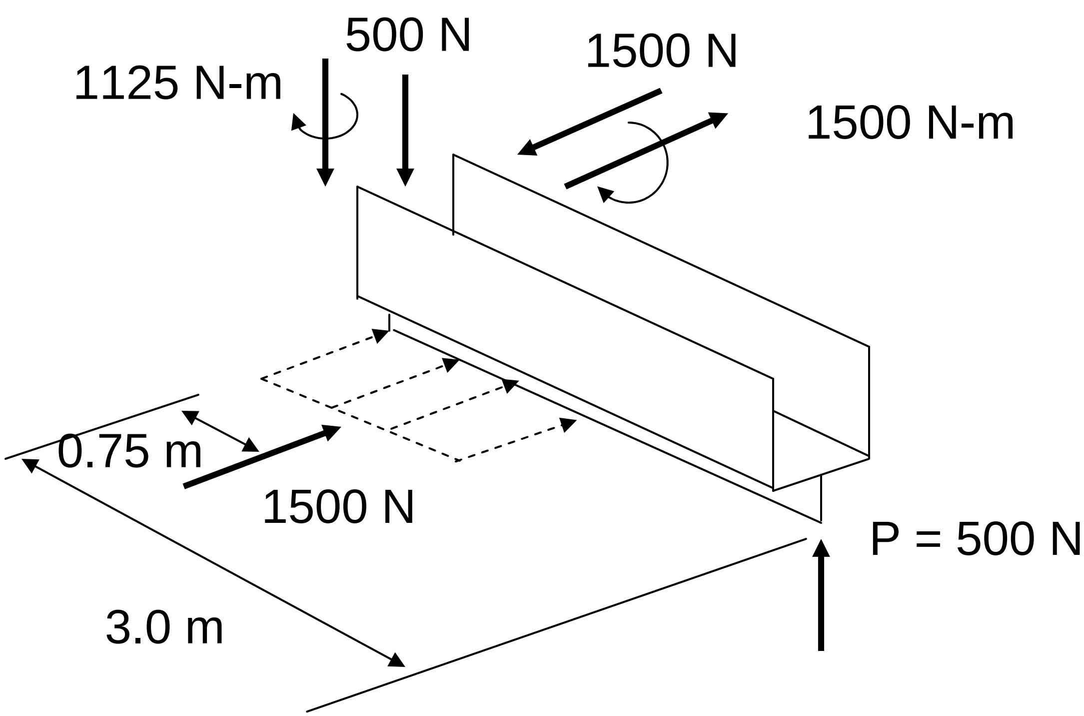
$$\delta = \sqrt{w^2 + v^2} = 40.8\ mm$$

Example 2

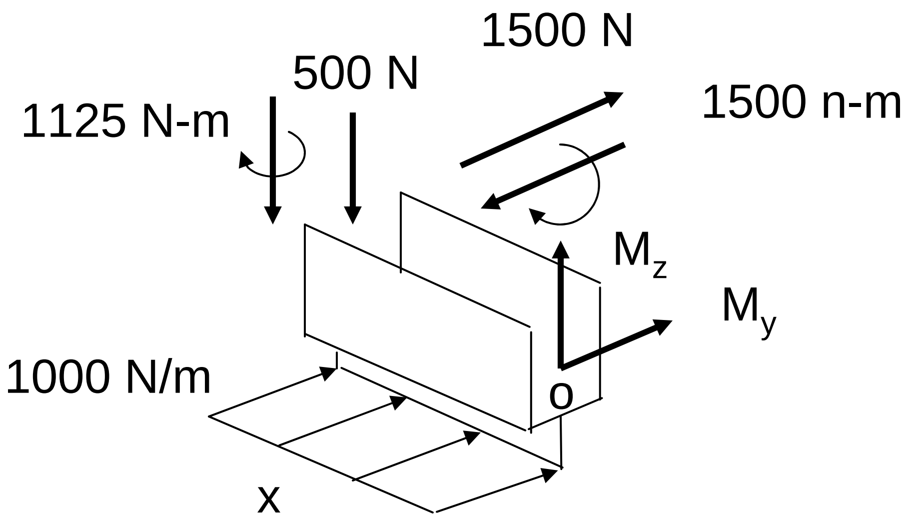


Determine the angle that the neutral surface makes with respect to the y-axis as a function of x. Use the cross section properties of example 1.

First, we need the reactions at the wall



For 0 < x < 1.5



$$\sum M_{zo} = 0$$

$$M_z + 1500x - 1000x^2/2 - 1125 = 0$$

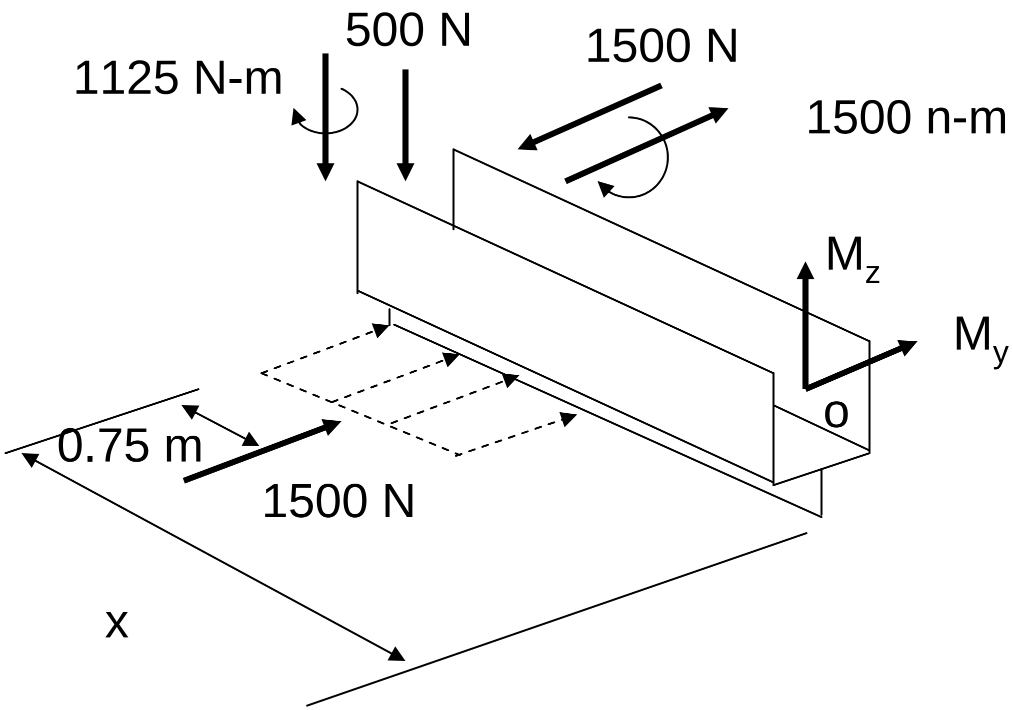
$$M_z = 500x^2 - 1500x + 1125 \ N - m$$

$$\sum M_{yo} = 0$$

$$M_y - 550x + 1500 = 0$$

$$M_y = -1500 + 500x \ N - m$$

For 1.5 < x < 3



$$\sum M_{zo} = 0$$

$$M_z + 1500x - 1500(x - 0.75) - 1125 = 0$$

$$M_z = 0$$

$$\sum M_{yo} = 0$$

$$M_y + 1500 - 500x = 0$$

$$M_y = 500x - 1500 \ N - m$$

$$\tan\lambda = \frac{I_{yy}}{I_{zz}}\frac{M_z}{M_y} = \frac{39.69}{30.73}\frac{M_z}{M_y} = 1.29\frac{M_z}{M_y}$$

0 < x < 1.5

$$\tan\lambda = 1.29\left(\frac{500x^2 - 1500x + 1125}{500x - 1500}\right)$$

1.5 < x < 3

$$\tan\lambda = 1.29\left(\frac{0}{500x - 1500}\right) = 0$$

$$\lambda = 0$$

```
% script neutral_axis

x=linspace(0,1.5,200);
num=1.29.*(500*x.^2-1500.*x+1125);
denom = 500.*x-1500;
ang=atan(num./denom);
ang = ang*180/pi; % change angle to degrees
x_plot=linspace(0,3,400);
ang_plot=[ang zeros(size(x))];
plot(x_plot, ang_plot)
xlabel('x-distance')
ylabel('angle, degrees')
```