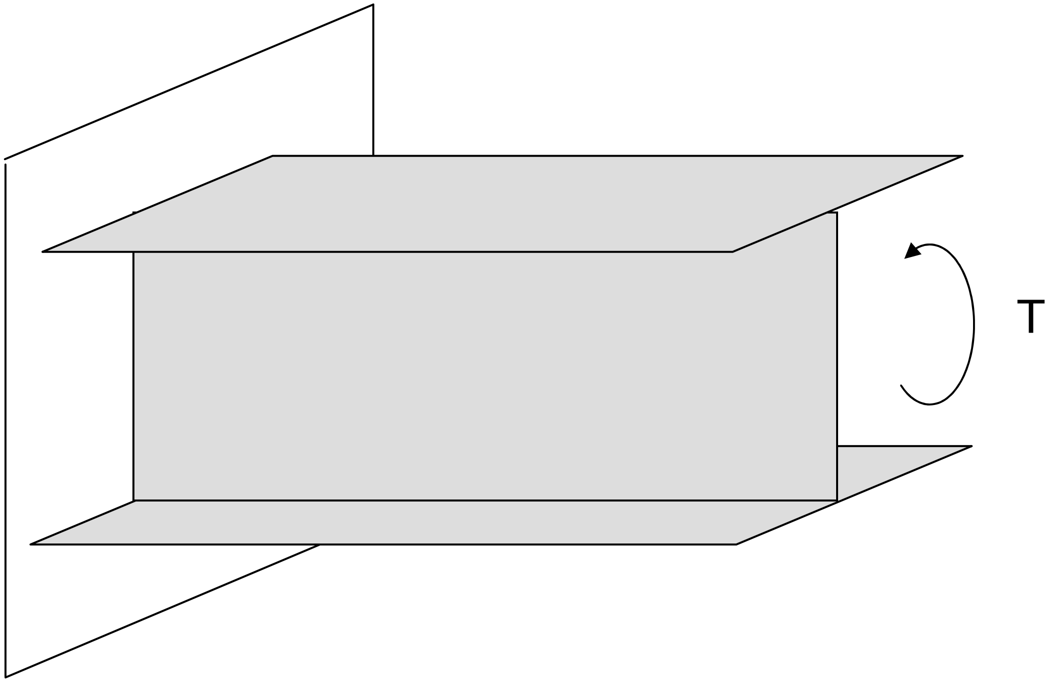
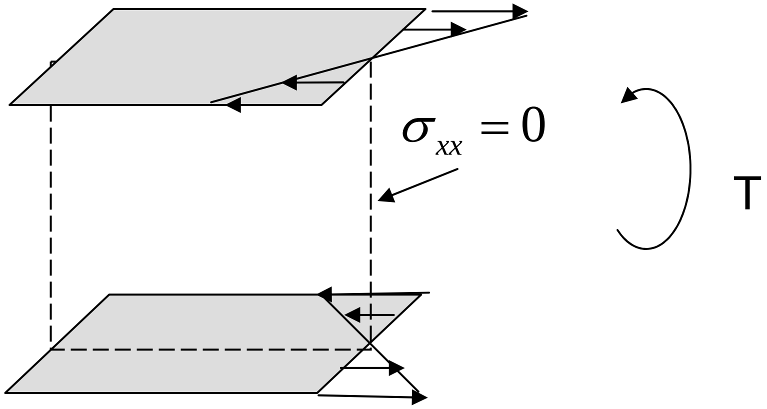
Twisting Induced by Axial Loads (Bi-Moments)



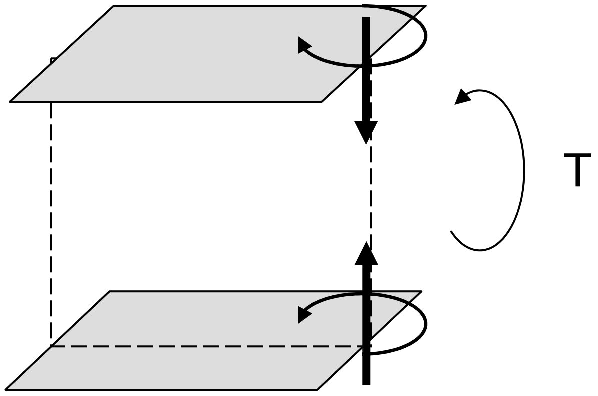
Recall, for torsion of a beam we found at the wall

$$\sigma_{xx} = -E\omega_p \frac{Tk}{GJ_{eff}} \tan(kl)$$

Since the stress is proportional to the principal sectorial area function, for the I-beam this normal stress distribution looks like:

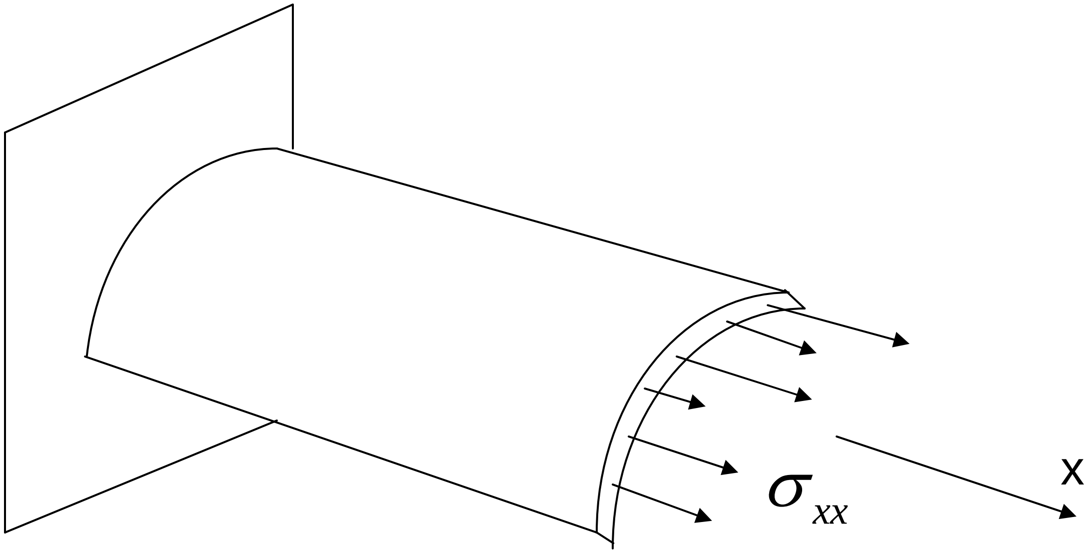


Thus, we see that the stresses in the flanges produce a self-equilibrated set of moments, called bi-moments:



If torsion (twisting) can generate bi-moments which are self-equilibrated axial stresses, then axial stress distributions that generate bi-moments should induce twisting

Thus, consider a thin, open section subjected to axial loads that generate both axial extension (or shortening) and twisting:



Then the axial displacement, by superposition is

$$u_x = -\omega_p\left(y,z\right)\beta\left(x\right) + u_o\left(x\right)$$

If we assume σ_{xx} is the only significant normal stress, then

$$\sigma_{xx} = E\frac{du_x}{dx} = -E\omega_p\frac{d\beta}{dx} + E\frac{du_0}{dx}$$

Let P = the axial load. Then

$$P = \int_A \sigma_{xx}\,dA = -E\frac{d\beta}{dx}\overbrace{\int_A \omega_p\,dA}^{=\,0} + EA\frac{du_0}{dx}$$

so we have

$$\frac{du_0}{dx} = \frac{P}{AE}$$

← just our usual axial load relationship

Also note that

$$\int_A y\sigma_{xx} dA = -E\frac{d\beta}{dx}\int_A y\omega_p dA + E\frac{du_0}{dx}\int_A y dA = 0$$

if y is measured from the centroid since then

$$\int_A y\omega_p dA = o$$

$$\int_A y dA = 0$$

Similarly

$$\int_A z\sigma_{xx} dA = -E\frac{d\beta}{dx}\int_A z\omega_p dA + E\frac{du_0}{dx}\int_A z dA = 0$$

if z is also measured from the centroid.

Thus, these axial stresses do not generate bending, only extension and twisting

Now, define the bi-moment, M_Γ , as

$$\boxed{M_\Gamma = \int_A \sigma_{xx}\omega_p dA}$$

Then

$$M_\Gamma = -E\frac{d\beta}{dx}\underbrace{\int_A \omega_p^2 dA}_{J_\omega} + e\frac{du_0}{dx}\cancelto{= 0}{\int_A \omega_p dA}$$

$$M_\Gamma = -E\,J_\omega \frac{d\beta}{dx}$$

and our stress

$$\sigma_{xx} = -E\omega_p \frac{d\beta}{dx} + E\frac{du_0}{dx}$$

becomes

$$\boxed{\sigma_{xx} = \frac{M_\Gamma \omega_p\left(y,z\right)}{J_\omega} + \frac{P}{A}}$$

Note: if the stress distribution is purely uniform (constant) over the cross section then no twisting will be induced since

$$M_\Gamma = \int_A \sigma_{xx}\omega_p dA = \sigma_{xx}\int_A \omega_p dA = 0$$

However, other axial stress distributions that generate a bi-moment will induce twisting (and extension)

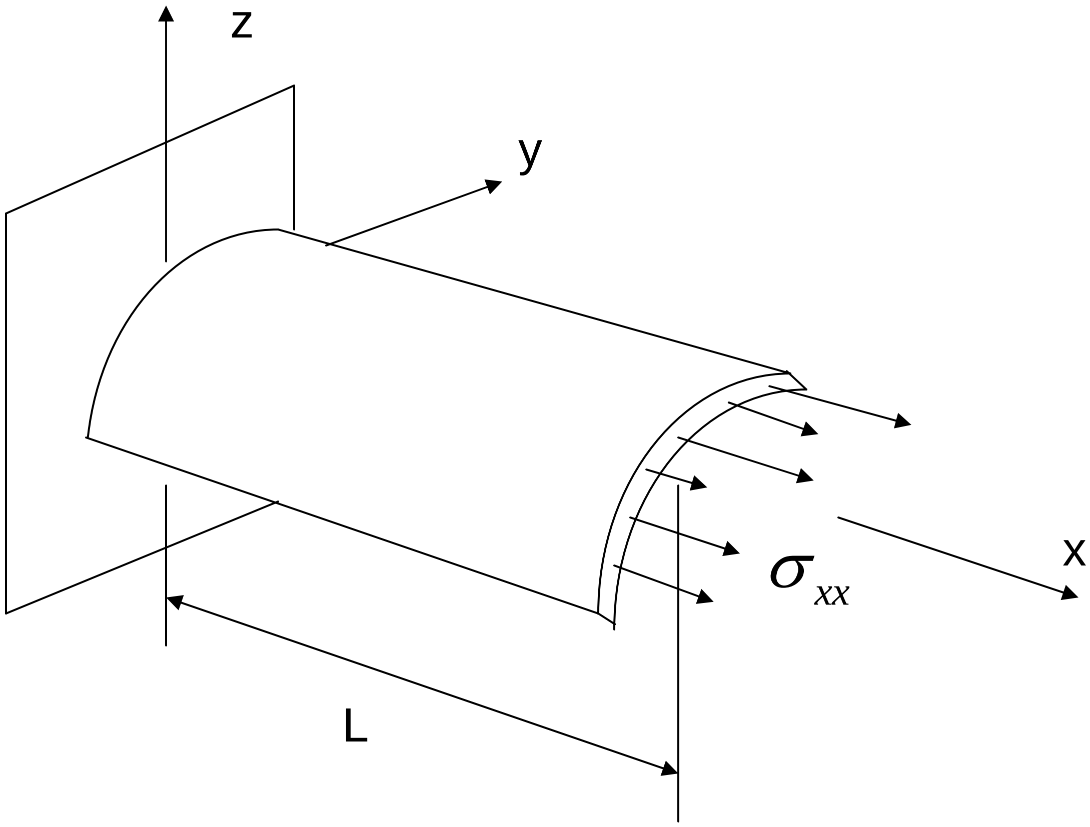
If σ_{xx} generates a bi-moment, how do we find the twisting, $\beta(x)$, this stress generates?

Answer: consider the relationship

$$\frac{d\beta}{dx} = -\frac{M_\Gamma}{EJ_\omega}$$

as a boundary condition on the end where the load is applied

For example:



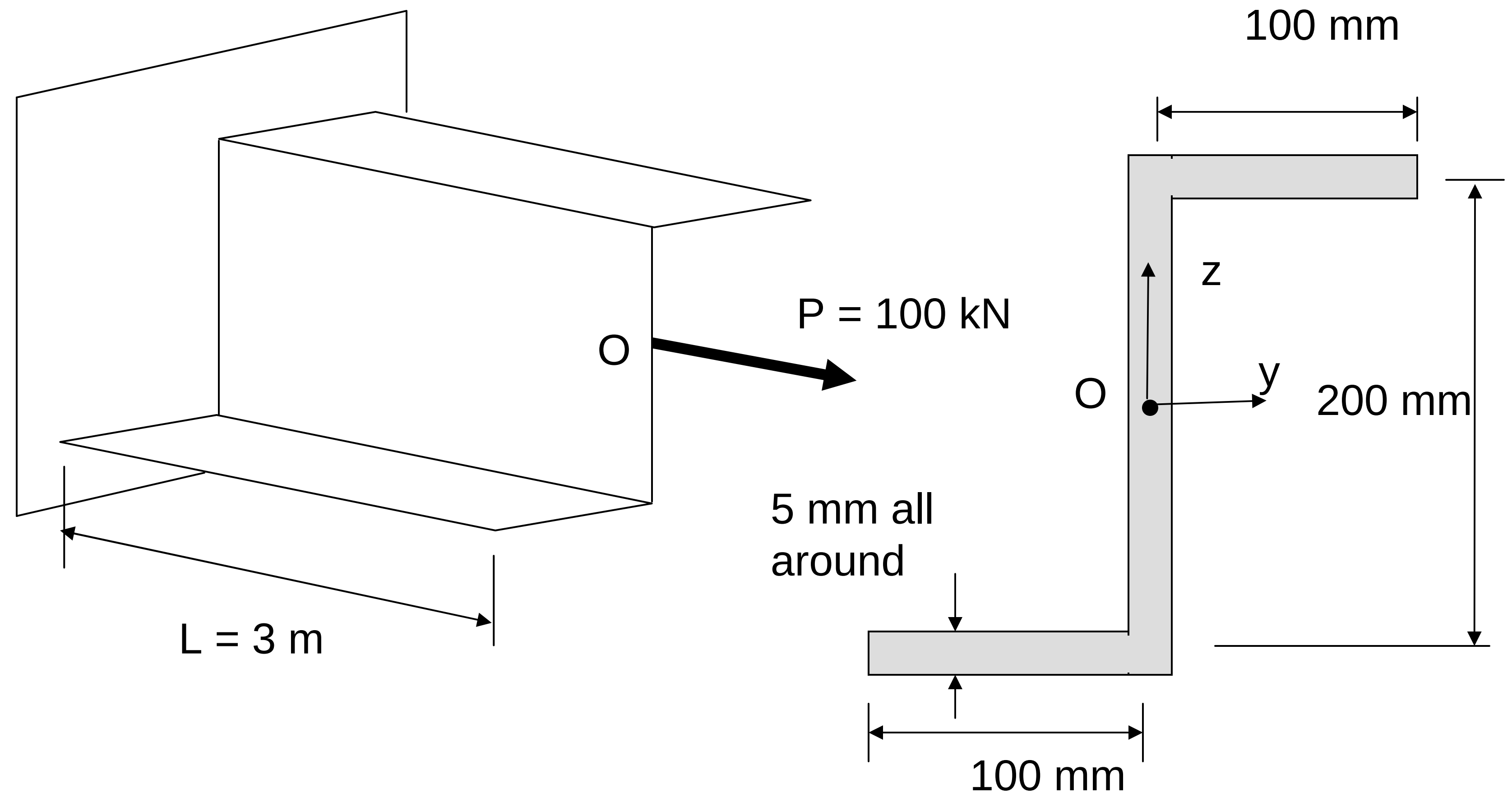
$$\frac{d^2\beta}{dx^2} - k^2\beta = 0$$ (since T = 0)

boundary conditions

$$\beta(0) = 0$$

$$\frac{d\beta}{dx}(L) = \frac{-M_\Gamma}{E\,J_\omega}$$

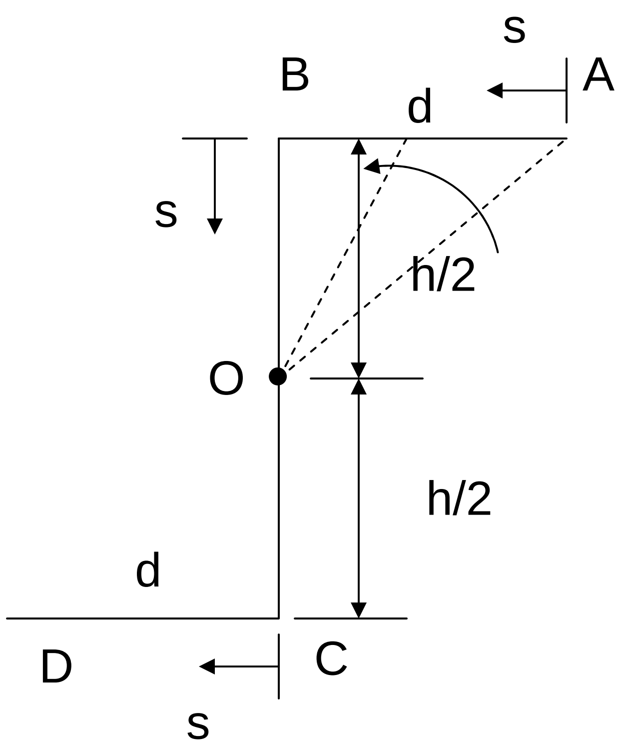
Consider a specific case of the cross-section shown below loaded by a concentrated axial force, P = 100 kN, acting through the centroid, O. Determine the twist, $\phi(x)$, and the stress distribution at the wall.



Note: O is also the shear center

$$E = 200\ GPa = 2\times10^5\ N/mm^2$$

$$G/E = 0.36$$



Take initial integration point at A with $\omega = \omega_0$

for AB $\quad \omega = \omega_0 + hs/2$

for BC $\quad \omega = \omega_0 + hd/2$

for CD $\quad \omega = \left(\omega_0 + hd/2\right) - hs/2$

Setting $\displaystyle\int_A \omega \, dA = 0$

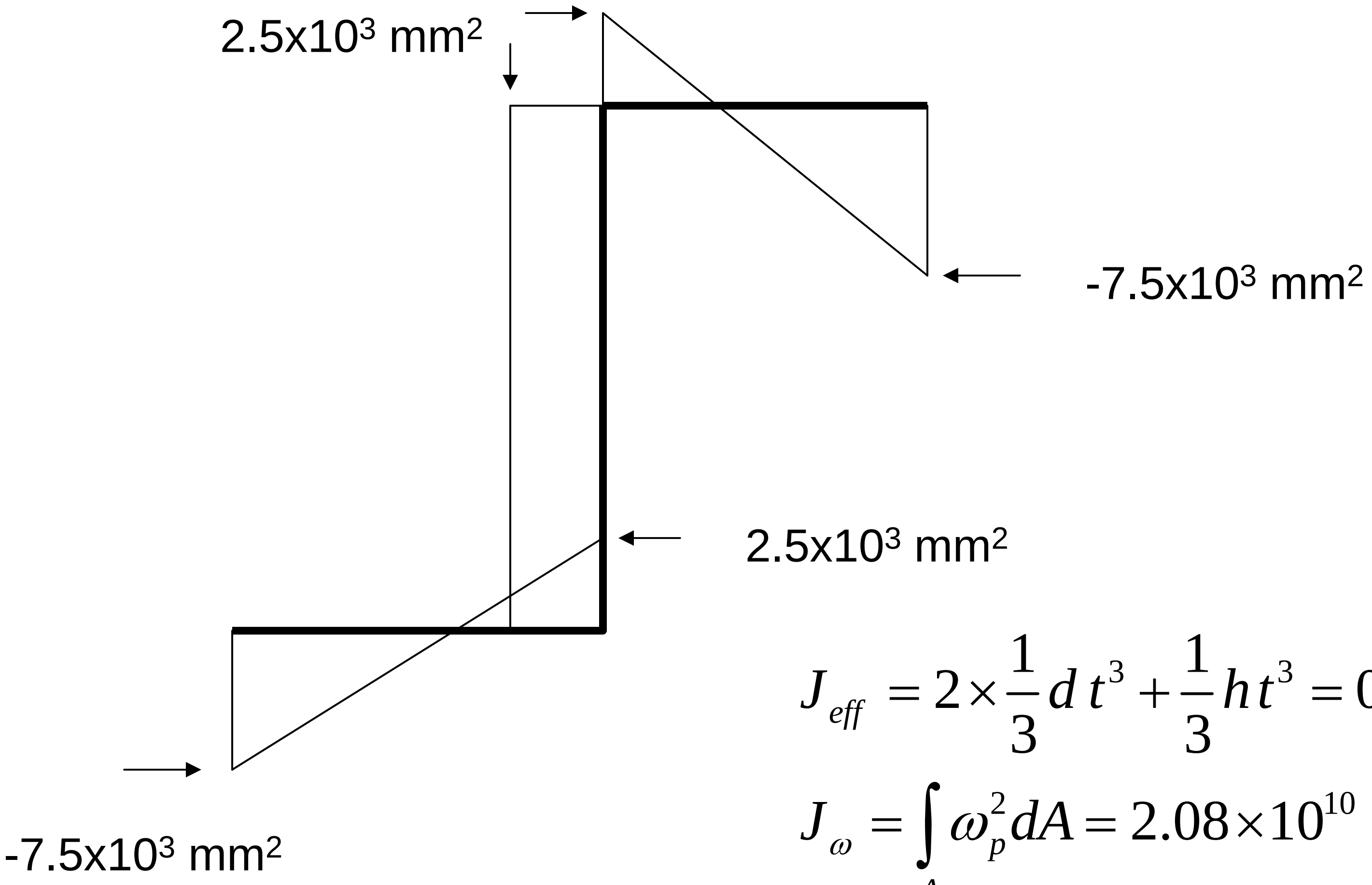
gives

$$t\int_0^d \left(\omega_0 + hs/2\right) ds + t\int_0^h \left(\omega_0 + hd/2\right) ds + t\int_0^d \left(\omega_0 + hd/2 - hs/2\right) ds = 0$$

$$\Rightarrow \quad \omega_0 = \frac{-hd}{2}\frac{\left(h+d\right)}{\left(h+2d\right)} = \frac{-\left(200\right)\left(100\right)}{2}\frac{\left(200+100\right)}{\left(200+200\right)}$$

$$= -7.5\times 10^3 \ mm^2$$

principal sectorial area function, ω_p



$$J_{eff} = 2\times\frac{1}{3}d\,t^3 + \frac{1}{3}h\,t^3 = 0.17\times10^5\ mm^4$$

$$J_\omega = \int_A \omega_p^2 dA = 2.08\times10^{10}\ mm^6$$

$$k = \sqrt{\frac{G\,J_{eff}}{E\,J_\omega}} = 5.4\times10^{-4}\ mm^{-1}$$

Since T = 0

$$\frac{d^2\beta}{dx^2} - k^2\beta = 0$$

$$\beta(x) = A\sinh(kx) + B\cosh(kx)$$

Boundary conditions

$$u_x(0) = 0 \rightarrow \beta(0) = 0 \rightarrow B = 0$$

$$\frac{d\beta(L)}{dx} = \frac{-M_\Gamma(L)}{E\,J_\omega}$$

⇨

$$Ak\cosh(kL) = \frac{-M_\Gamma(L)}{E\,J_\omega}$$

$$A = \frac{-M_\Gamma(L)}{E\,k\,J_\omega\cosh(kL)}$$

$$
\begin{aligned}
M_\Gamma(L) &= \int_A \sigma_{xx}(y,z)\,\omega_p(y,z)\,dA \\
&= \omega_p(0,0) \int_{A(0,0)} \sigma_{xx}\,dA \\
&= P\omega_p(0,0) = 2.5\times 10^3 P \quad N-mm
\end{aligned}
$$

$$
\begin{aligned}
A &= \frac{-M_\Gamma(L)}{E\,k\,J_\omega \cosh(kL)} \\
&= \frac{-2.5\times 10^3 P}{EkJ_\omega \cosh(kL)} = -4.23\times 10^{-5}\ mm^{-1}
\end{aligned}
$$

Thus,

$$
\beta(x) = -4.23\times 10^{-5} \sinh\left(5.4\times 10^{-4} x\right)\ mm^{-1}
$$

Integrating $\beta = \frac{d\phi}{dx}$ $\phi(x) = \frac{A}{k}\cosh(kx) + C$

Boundary condition $\phi(0) = 0 \rightarrow C = -A/k$

$$\phi(x) = \frac{A}{k}\left[\cosh(kx) - 1\right] \quad rad$$

$$= -0.08\left[\cosh\left(5.4\times10^{-4}x\right) - 1\right] \quad rad$$

at the free end

$$\phi(L) = -0.13 \quad rad$$

At the fixed wall

$$M_\Gamma(0) = -\frac{d\beta}{dx}(0)\,E\,J_\omega$$

$$= 95\times10^{6}\ N-mm^{2}$$

$$\sigma_{xx} = \frac{P}{A} + \frac{M_\Gamma \omega_p}{J_\omega}$$

N-mm^2 mm^2

$$= \frac{10^5}{(400)(5)} + \frac{\left(95\times10^{6}\right)\omega_p}{2.08\times10^{10}}$$

mm^6

$$= 50 + 4.57\times10^{-3}\,\omega_p \quad MPa$$