Recall, for torsion of a beam we found at the wall

\[ \sigma_{xx} = -E \omega_p \frac{Tk}{GJ_{eff}} \tan(kl) \]
Since the stress is proportional to the principal sectorial area function, for the I-beam this normal stress distribution looks like:

\[ \sigma_{xx} = 0 \]

Thus, we see that the stresses in the flanges produce a self-equilibrated set of moments, called bi-moments:
If torsion (twisting) can generate bi-moments which are self-equilibrated axial stresses, then axial stress distributions that generate bi-moments should induce twisting.

Thus, consider a thin, open section subjected to axial loads that generate both axial extension (or shortening) and twisting:
Then the axial displacement, by superposition is

$$u_x = -\omega_p (y, z) \beta (x) + u_o (x)$$

If we assume $\sigma_{xx}$ is the only significant normal stress, then

$$\sigma_{xx} = E \frac{du_x}{dx} = -E \omega_p \frac{d\beta}{dx} + E \frac{du_0}{dx}$$

Let $P =$ the axial load. Then

$$P = \int_A \sigma_{xx} dA = -E \frac{d\beta}{dx} \int_A \omega_p dA + EA \frac{du_0}{dx}$$

so we have

$$\frac{du_0}{dx} = \frac{P}{AE}$$

just our usual axial load relationship
Also note that

\[
\int_A y\sigma_{xx}dA = -E \frac{d\beta}{dx} \int_A y\omega_p dA + E \frac{du_0}{dx} \int_A ydA = 0
\]

if \( y \) is measured from the centroid since then

\[
\int_A y\omega_p dA = 0
\]

\[
\int_A ydA = 0
\]

Similarly

\[
\int_A z\sigma_{xx}dA = -E \frac{d\beta}{dx} \int_A z\omega_p dA + E \frac{du_0}{dx} \int_A zdA = 0
\]

if \( z \) is also measured from the centroid.

Thus, these axial stresses do not generate bending, only extension and twisting.
Now, define the bi-moment, $M_\Gamma$, as

$$M_\Gamma = \int_A \sigma_{xx} \omega_p dA$$

Then

$$M_\Gamma = -E \frac{d \beta}{dx} \int_A \omega_p^2 dA + e \frac{du_0}{dx} \int_A \omega_p dA$$

$$M_\Gamma = -E J_\omega \frac{d \beta}{dx}$$

and our stress

$$\sigma_{xx} = -E \omega_p \frac{d \beta}{dx} + E \frac{du_0}{dx}$$

becomes

$$\sigma_{xx} = \frac{M_\Gamma \omega_p (y, z)}{J_\omega} + \frac{P}{A}$$
Note: if the stress distribution is purely uniform (constant) over the cross section then no twisting will be induced since

$$M_\Gamma = \int_A \sigma_{xx} \omega_p \, dA = \sigma_{xx} \int_A \omega_p \, dA = 0$$

However, other axial stress distributions that generate a bi-moment will induce twisting (and extension)

If $\sigma_{xx}$ generates a bi-moment, how do we find the twisting, $\beta(x)$, this stress generates?

Answer: consider the relationship

$$\frac{d \beta}{dx} = - \frac{M_\Gamma}{E J_\omega}$$

as a boundary condition on the end where the load is applied
For example:

\[
\frac{d^2 \beta}{dx^2} - k^2 \beta = 0 \quad \text{(since } T = 0)\]

boundary conditions

\[
\beta(0) = 0
\]

\[
\frac{d \beta}{dx}(L) = \frac{-M_\Gamma}{E J_\omega}
\]
Consider a specific case of the cross-section shown below loaded by a concentrated axial force, \( P = 100 \text{ kN} \), acting through the centroid, \( O \). Determine the twist, \( \phi(x) \), and the stress distribution at the wall.

Note: \( O \) is also the shear center

\[
E = 200 \text{ GPa} = 2 \times 10^5 \text{ N/mm}^2
\]

\[
G / E = 0.36
\]
Take initial integration point at A with $\omega = \omega_0$

For AB

$$\omega = \omega_0 + hs / 2$$

For BC

$$\omega = \omega_0 + hd / 2$$

For CD

$$\omega = (\omega_0 + hd / 2) - hs / 2$$

Setting $\int_A \omega \, dA = 0$

gives

$$t \int_0^d (\omega_0 + hs / 2) \, ds + t \int_0^h (\omega_0 + hd / 2) \, ds + t \int_0^d (\omega_0 + hd / 2 - hs / 2) \, ds = 0$$

$$\omega_0 = -\frac{hd}{2} \frac{(h + d)}{(h + 2d)} = -\frac{(200)(100)}{2} \frac{(200 + 100)}{(200 + 200)}$$

$$= -7.5 \times 10^3 \text{ mm}^2$$
principal sectorial area function, $\omega_p$

$$J_{\text{eff}} = 2 \times \frac{1}{3} d t^3 + \frac{1}{3} h t^3 = 0.17 \times 10^5 \text{ mm}^4$$

$$J_\omega = \int_A \omega_p^2 dA = 2.08 \times 10^{10} \text{ mm}^6$$

$$k = \sqrt{\frac{G J_{\text{eff}}}{E J_\omega}} = 5.4 \times 10^{-4} \text{ mm}^{-1}$$
Since $T = 0$

$$\frac{d^2 \beta}{dx^2} - k^2 \beta = 0$$

$$\beta(x) = A \sinh(kx) + B \cosh(kx)$$

Boundary conditions

$$u_x(0) = 0 \rightarrow \beta(0) = 0 \rightarrow B = 0$$

$$\frac{d \beta(L)}{dx} = -\frac{M_{\Gamma}(L)}{E J_\omega}$$

$$Ak \cosh(kL) = -\frac{M_{\Gamma}(L)}{E J_\omega}$$

$$A = \frac{-M_{\Gamma}(L)}{E k J_\omega \cosh(kL)}$$
\[
M_{\Gamma}(L) = \int_A \sigma_{xx}(y, z) \omega_p(y, z) \, dA \\
= \omega_p(0, 0) \int_{A(0,0)} \sigma_{xx} \, dA \\
= P \omega_p(0, 0) = 2.5 \times 10^3 \, P \quad N - mm
\]

\[
A = \frac{-M_{\Gamma}(L)}{E k J_0 \cosh(kL)} \\
= \frac{-2.5 \times 10^3 P}{E k J_0 \cosh(kL)} = -4.23 \times 10^{-5} \, mm^{-1}
\]

Thus,
\[
\beta(x) = -4.23 \times 10^{-5} \sinh(5.4 \times 10^{-4} x) \, mm^{-1}
\]
Integrating \( \beta = \frac{d\phi}{dx} \quad \phi(x) = \frac{A}{k} \cosh(kx) + C \)

Boundary condition \( \phi(0) = 0 \rightarrow C = -\frac{A}{k} \)

\[
\phi(x) = \frac{A}{k} \left[ \cosh(kx) - 1 \right] \quad \text{rad}
\]

\[
= -0.08 \left[ \cosh \left( 5.4 \times 10^{-4} x \right) - 1 \right] \quad \text{rad}
\]

at the free end \( \phi(L) = -0.13 \quad \text{rad} \)
At the fixed wall

\[ M_\Gamma (0) = -\frac{d\beta}{dx}(0) E J_\omega \]
\[ = 95 \times 10^6 \text{ N} - \text{mm}^2 \]

\[ \sigma_{xx} = \frac{P}{A} + \frac{M_\Gamma \omega_p}{J_\omega} \text{ N-mm}^2 \text{ mm}^2 \]
\[ = \frac{10^5}{(400)(5)} + \frac{(95 \times 10^6) \omega_p}{2.08 \times 10^{10} \text{ mm}^6} \]
\[ = 50 + 4.57 \times 10^{-3} \omega_p \text{ MPa} \]