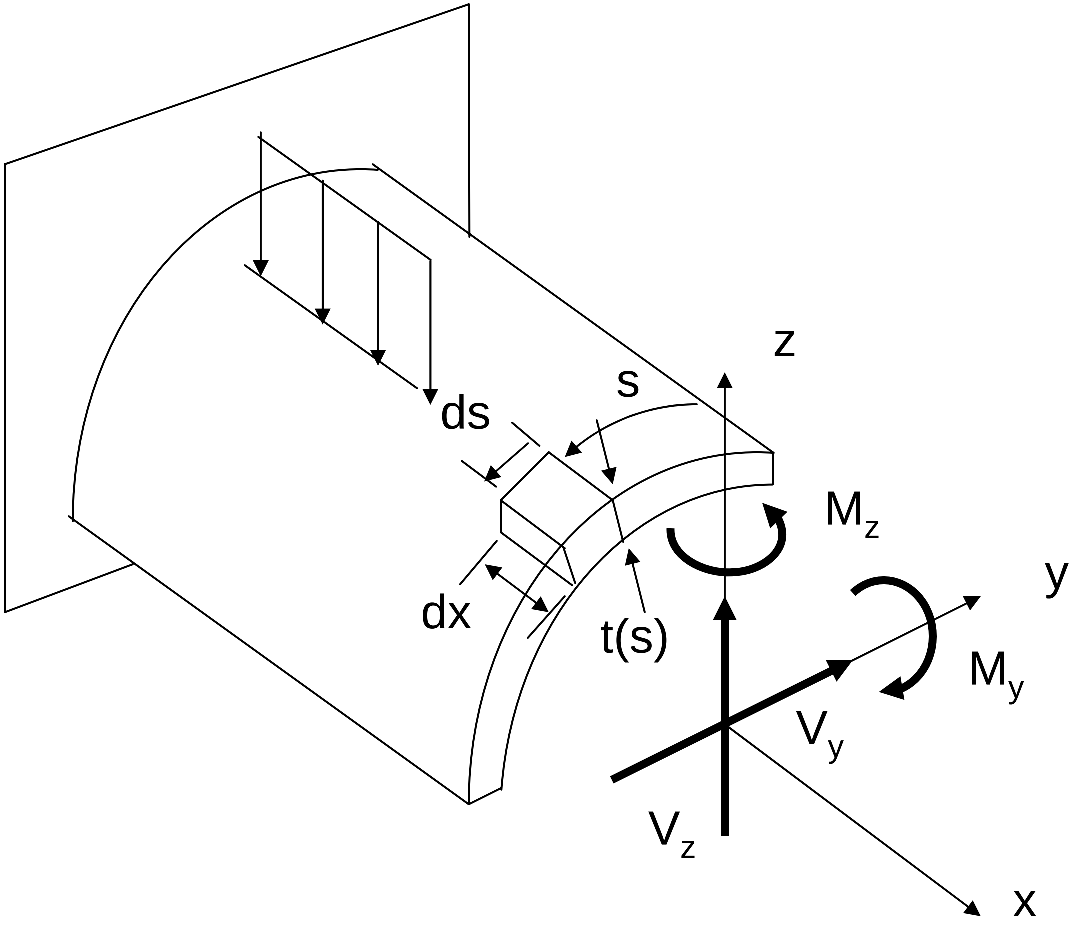
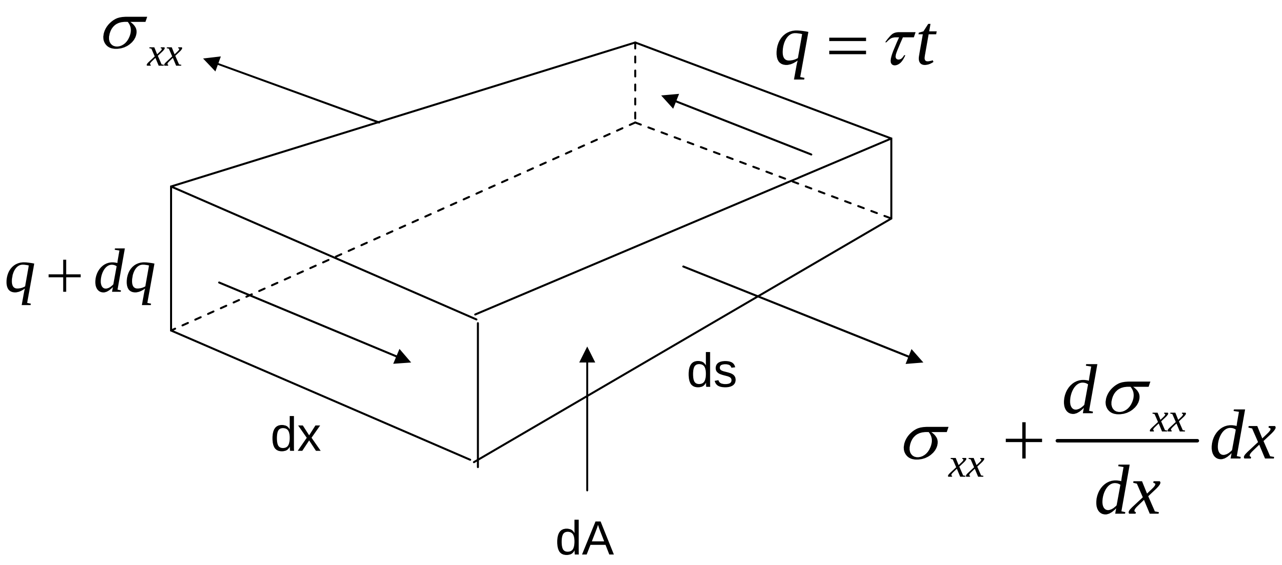
Shear stresses in the bending of thin, unsymmetrical sections



consider an element of a thin beam, as shown

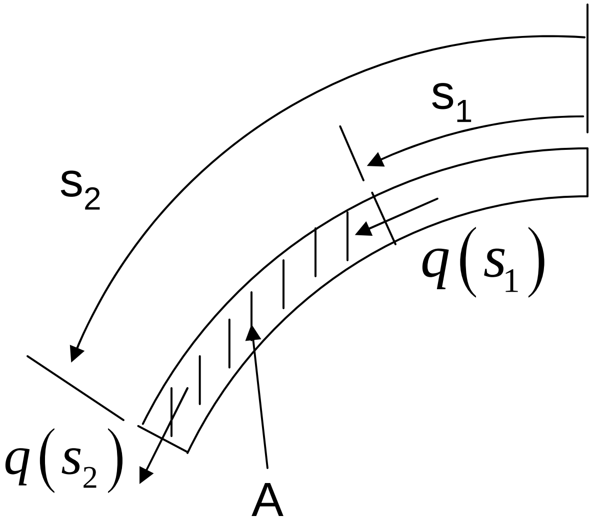


$$\sum F_x = 0$$

$$\left(q + dq\right) dx - q dx + \left(\sigma_{xx} + \frac{d\sigma_{xx}}{dx} dx\right) dA - \sigma_{xx} dA = 0$$

$$dq = -\frac{d\sigma_{xx}}{dx} dA$$

If we integrate from s_1 to s_2



$$\int_{s_1}^{s_2} dq = -\int_A \frac{d\sigma_{xx}}{dx} dA$$

$$q(s_2) - q(s_1) = \Delta q = -\int_A \frac{d\sigma_{xx}}{dx} dA$$

Since

$$\sigma_{xx} = \frac{\left(M_y I_{zz} + M_z I_{yz}\right) z - \left(M_z I_{yy} + M_y I_{yz}\right) y}{\left(I_{yy} I_{zz} - I_{yz}^2\right)}$$

$$\frac{d\sigma_{xx}}{dx} = \frac{\left(\overset{V_z}{\frac{dM_y}{dx}} I_{zz} + \overset{-V_y}{\frac{dM_z}{dx}} I_{yz}\right) z - \left(\overset{-V_y}{\frac{dM_z}{dx}} I_{yy} + \overset{V_z}{\frac{dM_y}{dx}} I_{yz}\right) y}{\left(I_{yy} I_{zz} - I_{yz}^2\right)}$$

$$\frac{d\sigma_{xx}}{dx} = \frac{\left(V_z I_{zz} - V_y I_{yz}\right) z - \left(V_z I_{yz} - V_y I_{yy}\right) y}{\left(I_{yy} I_{zz} - I_{yz}^2\right)}$$

placing this in our shear flow expression gives

$$\Delta q = \frac{\left(V_z I_{yz} - V_y I_{yy}\right) Q_z + \left(V_y I_{yz} - V_z I_{zz}\right) Q_y}{D}$$

where $$D = I_{yy} I_{zz} - I_{yz}^2$$

$$Q_z = \int_A y dA$$

$$Q_y = \int_A z dA$$

If we have a symmetrical expression and if $V_y = 0$

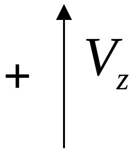
$$\Delta q = \frac{-V_z Q_y}{I_{yy}}$$

If q(s_1) =0 then

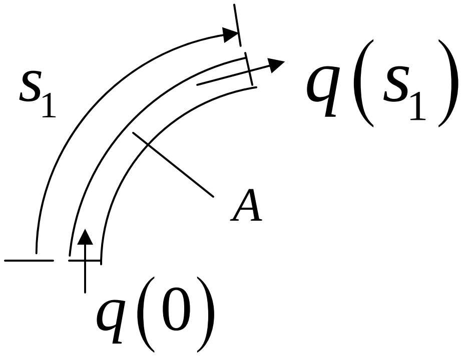
$$\Delta q = \tau\left(s_1\right) t\left(s_1\right)$$

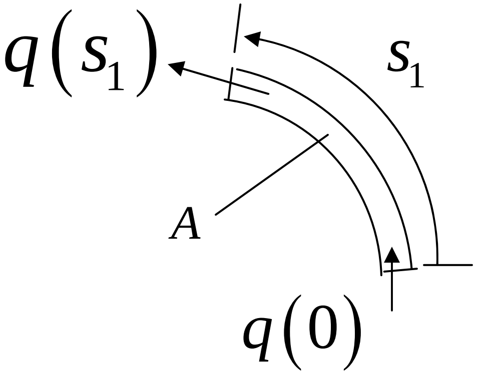
and we obtain
$$\tau = \frac{-V_z Q_y}{I_{yy}\, t}$$

the minus sign is here since we have taken



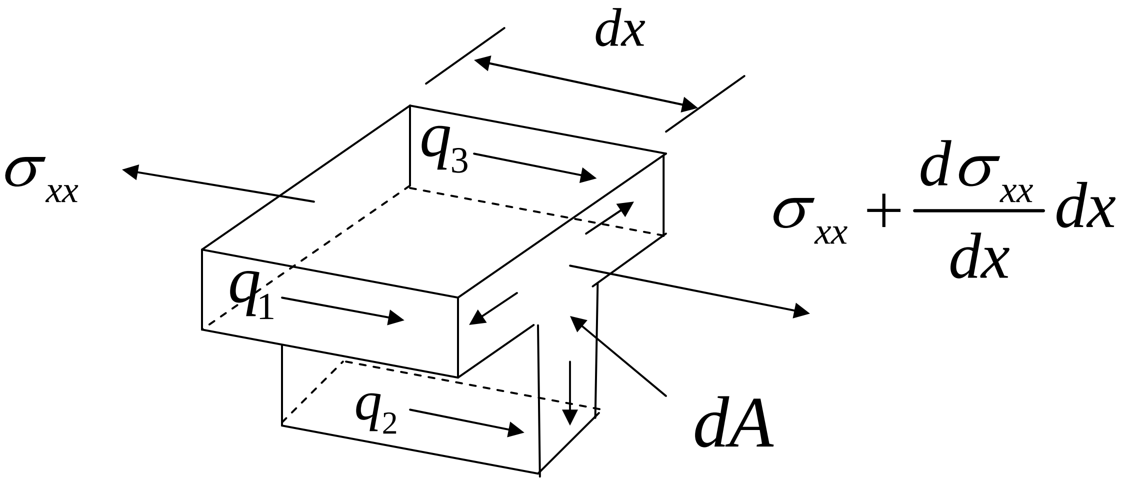
Note: in our expression for Δq the q flows "out" from the end of the section under consideration, whether the section is cut from the left or right of the cross section. At the beginning of the section the q flows "in". Examples:





in both these cases
$$\Delta q = q(s_1) - q(0) = \frac{\left(V_z I_{yz} - V_y I_{yy}\right) Q_z + \left(V_y I_{yz} - V_z I_{zz}\right) Q_y}{D}$$

Just as in the torsion of closed sections, the shear flows generated by bending must be conserved at a junction. This follows from equilibrium:



$$\sum F_x = 0$$

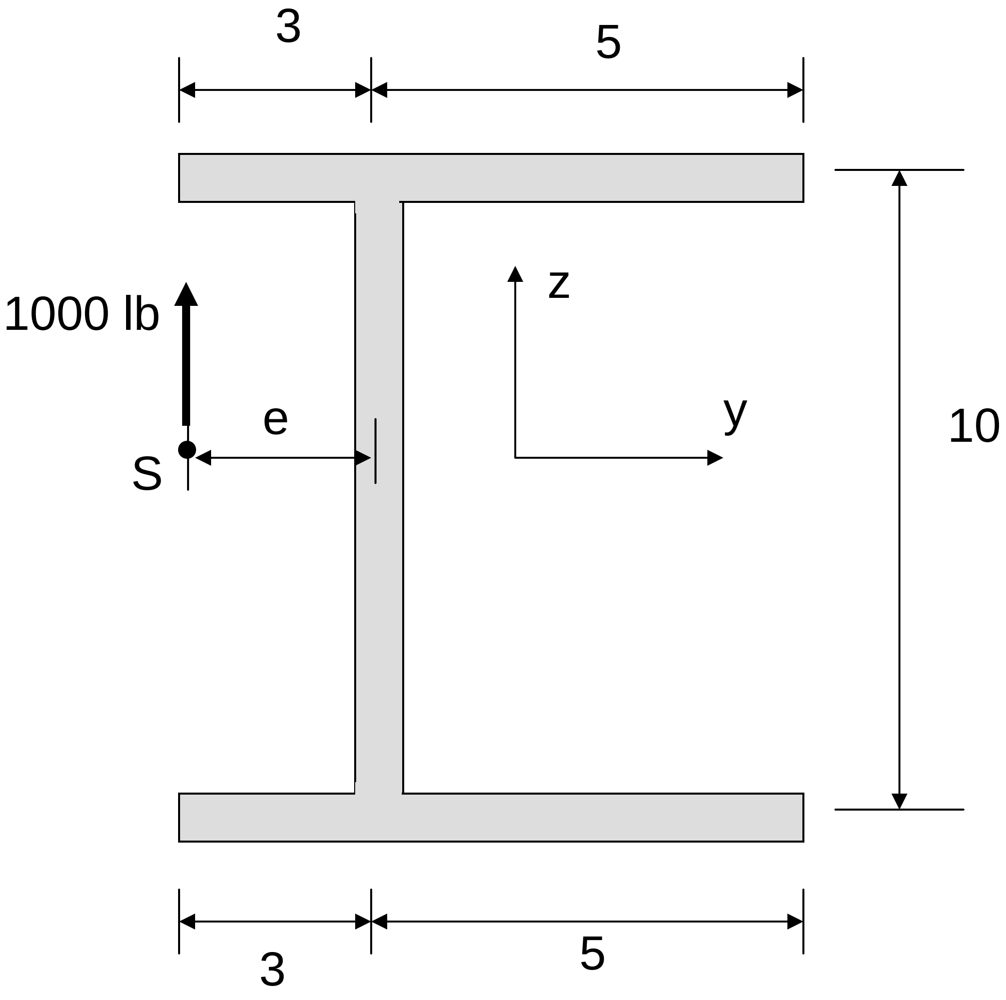
$$\left(q_1 + q_2 + q_3\right) dx + \left(\sigma_{xx} + \frac{d\sigma_{xx}}{dx} dx\right) dA - \sigma_{xx} dA = 0$$

$$\left(q_1 + q_2 + q_3\right) + d\sigma_{xx} \frac{dA}{dx} = 0$$

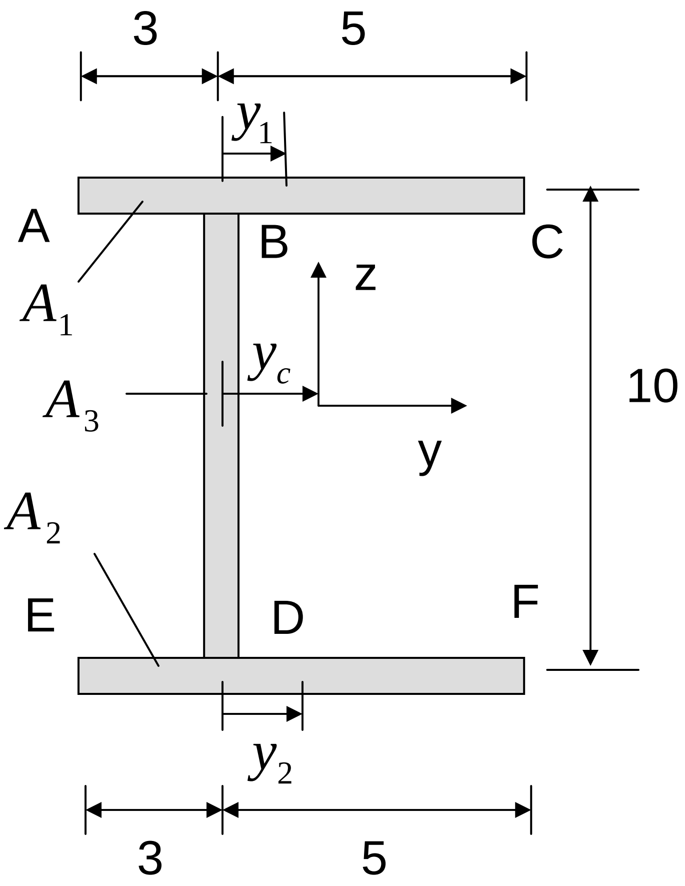
so as $dA \to 0$ $\quad q_1 + q_2 + q_3 = 0$

i.e. the net shear flow out of (or into) a junction must be zero

A 1000 lb load produces bending but no twisting in the thin open section shown below. Determine the shear flow distributions in this section and the location of the shear center. Assume the thickness of all sections is 0.1 in. All dimensions are measured to the mid planes of the walls.



all dimensions are in inches



$$y_c = \frac{2\times\left(y_1 A_1\right) + (0)\,A_3}{2A_1 + A_3}$$

$$= \frac{(2)(8)(0.1)(1)}{(2)(8)(0.1)+(10)(.1)}$$

$$= 0.6154\ in$$

$$V_z = 1000\ lb, \quad V_y = 0, \quad I_{yz} = 0$$

so we have

$$\Delta q = \frac{-V_z Q_y}{I_{yy}}$$

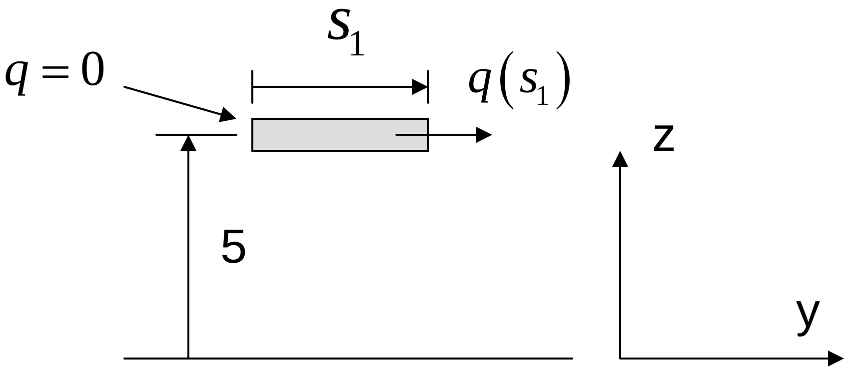
negligible

$$I_{yy} = 2\times\left[\frac{1}{12}(8)\cancel{(0.1)}^3 + (8)(0.1)(5)^2\right] + \frac{1}{12}(0.1)(10)^3$$

$$= 48.33\ in^4$$

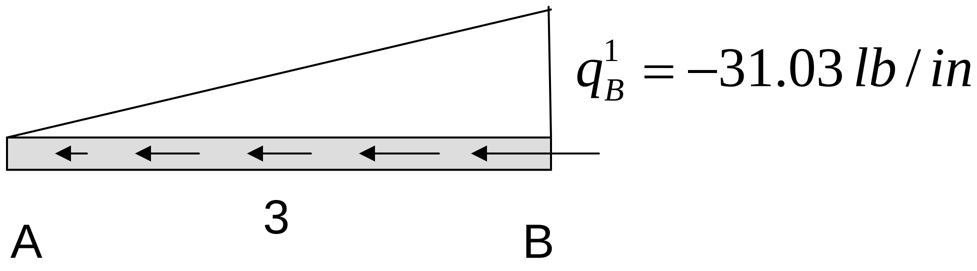
$$\Delta q = \frac{-1000}{48.33} Q_y = -20.69 Q_y$$

for AB

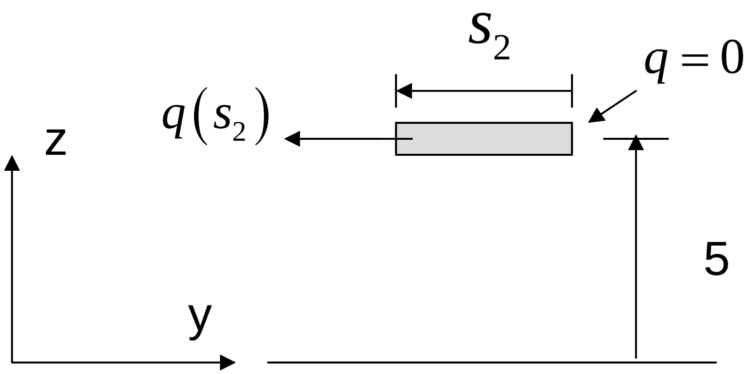


$$\Delta q = q(s_1) - 0 = -20.69\left[(0.1)(s_1)(5)\right]$$

$$q(s_1) = -10.35\, s_1 \quad lb/in$$



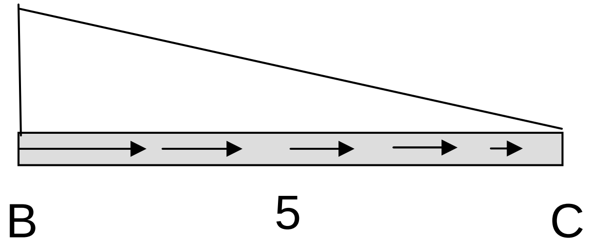
for BC



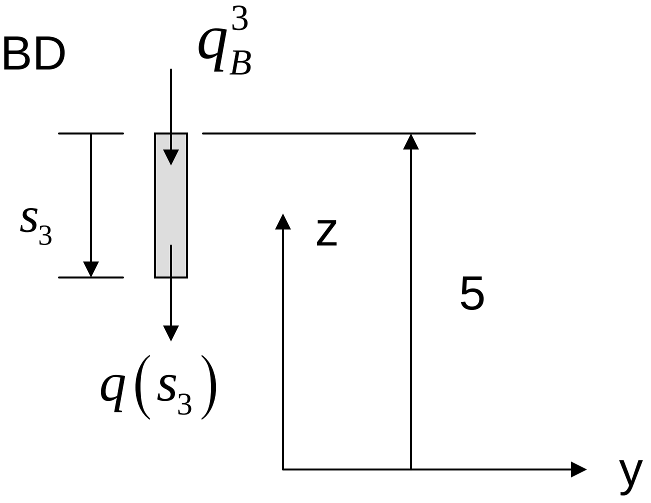
$$\Delta q = q(s_2) - 0 = -20.69\left[(0.1)(s_2)(5)\right]$$

$$q(s_2) = -10.35\, s_2$$

$$q_B^2 = -51.75\, lb/in$$

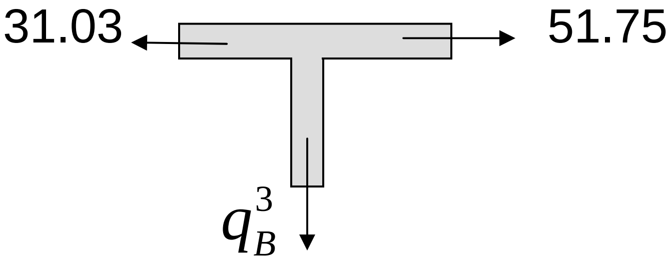


for BD



$$\Delta q = q(s_3) - q_B^3 = -20.69\left[(0.1)(s_3)(5 - s_3/2)\right]$$

$$q(s_3) = q_B^3 - 10.35\,s_3 + 1.035\,s_3^2$$

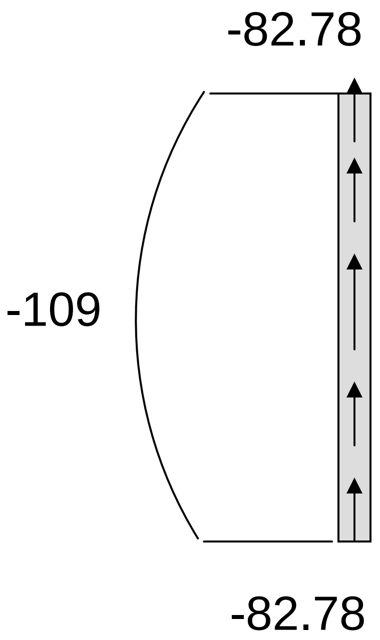


$$q_B^3 + 51.75 - 31.03 = 0$$

$$q_B^3 = -82.78\ \ lb/in$$

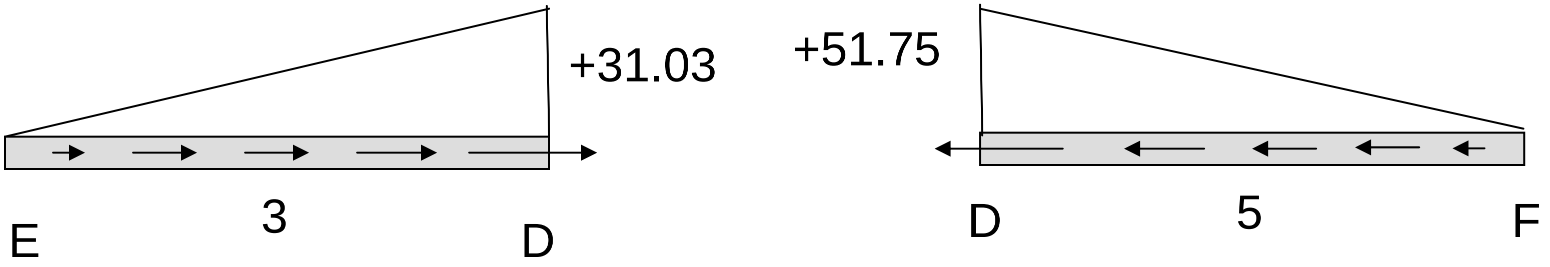
$$q(s_3) = -82.78 - 10.35\,s_3 + 1.035\,s_3^2$$

At the middle $q(5) = -109\ lb/in$

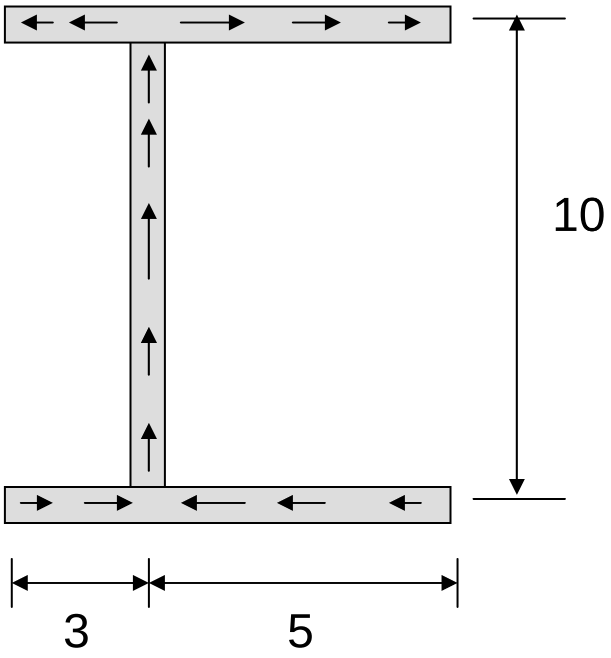


$$q(s_3) = -82.78 - 10.35\,s_3 + 1.035\,s_3^2$$

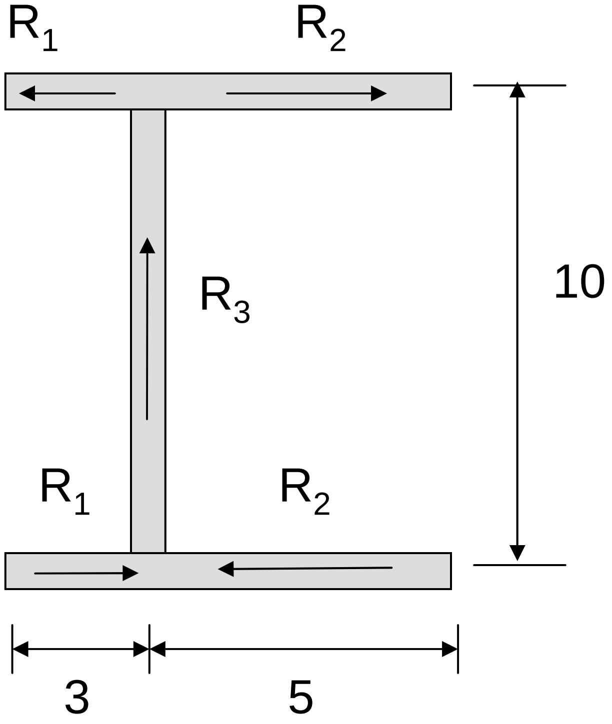
For the lower flange Q_y is the same as for the upper flange but with opposite sign so we have



the entire shear stress distribution:



the total forces carried by each section:

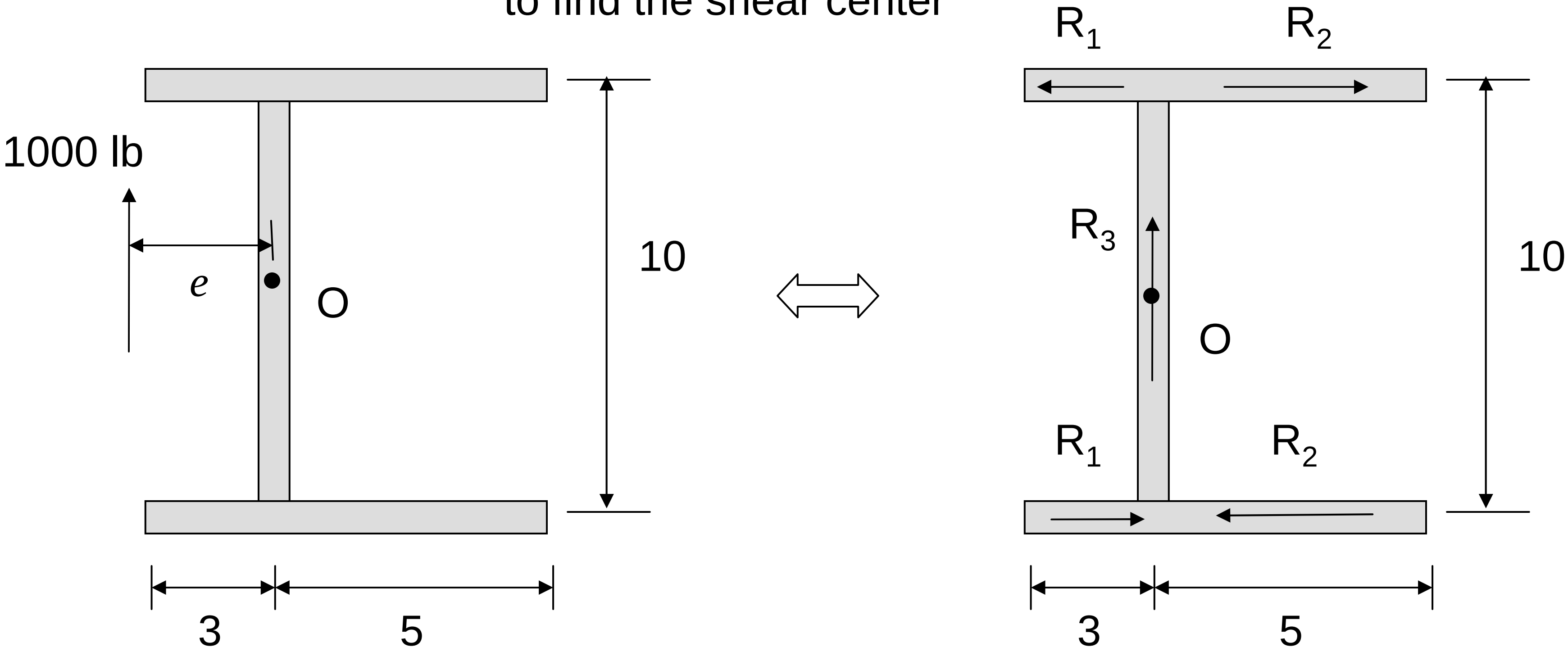


$$R_1 = \int_0^3 10.35 s_1 \, ds_1 = 46.58\, lb$$

$$R_2 = \int_0^5 10.35 s_2 \, ds_2 = 129.38\, lb$$

$$R_3 = \int_0^{10} \left(82.78 + 10.35 s_3 - 1.035 s_3^2 \right) ds_3 = 1000\, lb$$

Note: we have used the negative of the shear flow expressions to agree with the directions assumed for the forces shown to find the shear center



We obtained the shear flow distributions under the assumption that there was only bending. These shear flows will generate the shear forces and their moments about a particular point (i.e. the force locations). Thus, we can determine the shear center here by requiring that the moment of the 1000 lb load about any point produce the same moments as the shear flows,

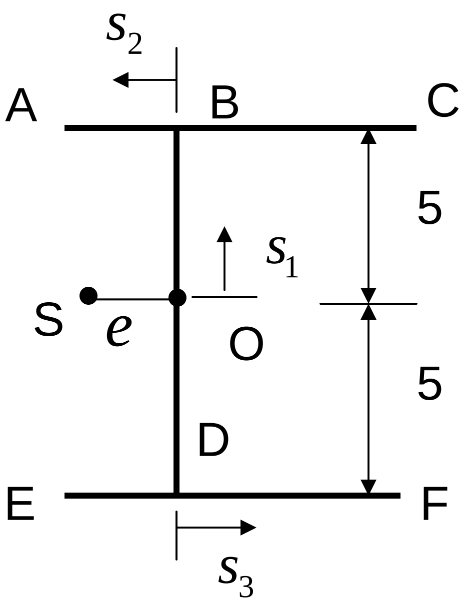
$$\sum M_O = 1000e \qquad \sum M_O = 10R_2 - 10R_1 = (129.38 - 46.58)(10)$$

$$= 828 \, in - lb$$

$$1000e = 828$$

$$e = 0.828 \, in$$

location of the shear center by sectorial area



take the starting point for the integration at O

BD $\omega = \omega_0 + e s_1$

AC $\omega = \omega_0 + 5e + 5s_2$

EF $\omega = \omega_0 - 5e + 5s_3$

$$\int \omega dA = 0$$

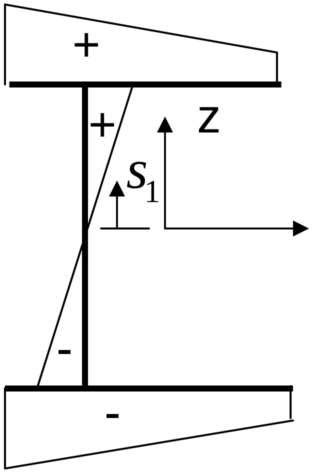
$$t\left[\int_{-5}^{5}\left(\omega_0 + \cancel{es_1}\right)ds_1 + \int_{-5}^{3}\left(\omega_0 + \cancel{5e} + \cancel{5s_2}\right)ds_2 + \int_{-3}^{5}\left(\omega_0 - \cancel{5e} + \cancel{5s_3}\right)ds_3\right] = 0$$

odd

$$10\omega_0 + 8\omega_0 + 8\omega_0 = 0$$

$$\Rightarrow \quad \boxed{\omega_0 = 0}$$

so the sectorial area distribution looks like:



Thus, $\int y\omega dA = 0$

automatically and we need only consider

$$\int z\omega dA = 0$$

$$\int z\omega dA = 0$$

$$t\left[\int_{-5}^{+5}\left(es_1\right)s_1 ds_1 + 5\int_{-5}^{3}\left(5e+5s_2\right)ds_2 + \left(-5\right)\int_{-3}^{5}\left(-5e+5s_3\right)ds_3\right] = 0$$

$$\uparrow \qquad\qquad \uparrow \qquad\qquad\qquad\qquad \uparrow$$

$$z = s_1 \qquad z = 5 \qquad\qquad\qquad z = -5$$

which gives $\dfrac{250}{3}e + 400e - 400 = 0$

$$\Rightarrow \qquad e = \frac{400}{400 + 250/3} = 0.828\,in$$