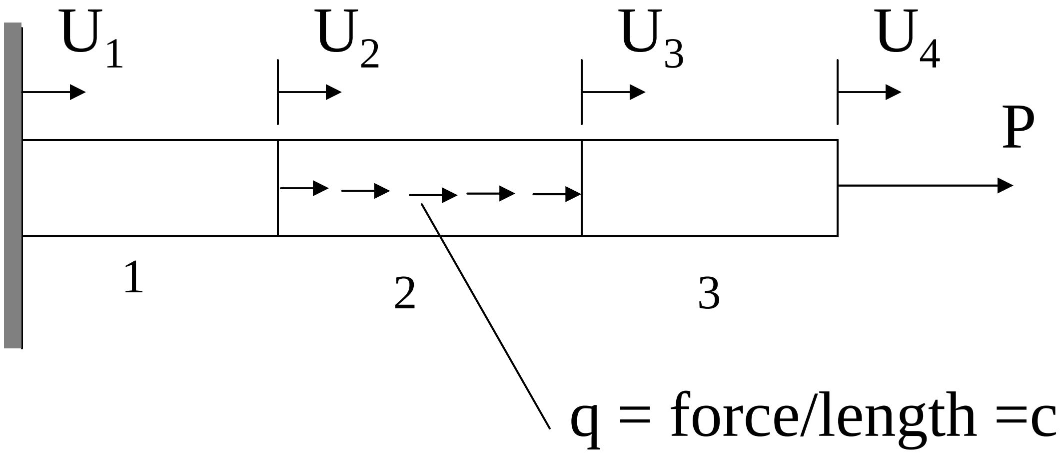
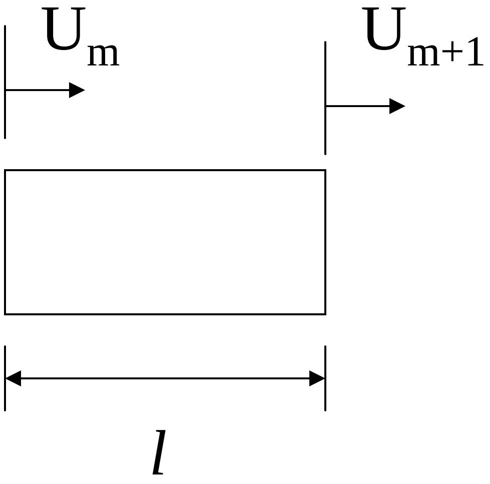
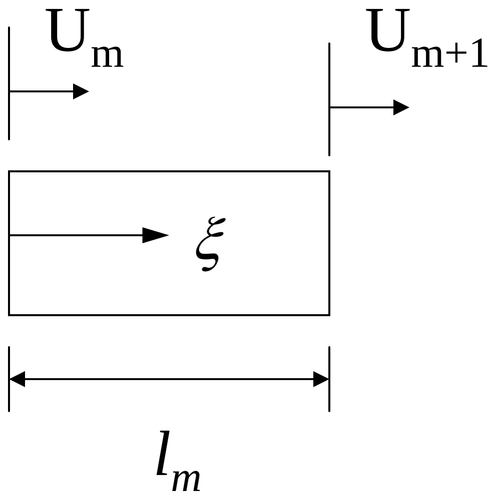
1-D Finite Elements – Virtual Work Approach



q = force/length =constant



mth element



1. Assume some variation of displacements in the element and write in terms of the nodal displacements

Simplest case: linear variation

$$u_x^{(m)} = U_m + \frac{\left(U_{m+1} - U_m\right)\xi}{l_m}$$

$$= U_m \underbrace{\left(1 - \frac{\xi}{l_m}\right)}_{H_m^{(m)}(\xi)} + U_{m+1} \underbrace{\left(\frac{\xi}{l_m}\right)}_{H_{m+1}^{(m)}(\xi)}$$

shape functions for displacement

If we let $H_j^{(m)} = 0 \quad j \neq m, m+1$ then we can write the displacement

in each element in terms of <u>all</u> the nodal variables

$$u_x^{(m)} = \sum_{j=1}^{M+1} H_j^{(m)} U_j \quad \left(m = 1, \ldots, M\right)$$

(if we have M elements
we have M+1 nodes)

2. Compute the strains for each element in terms of the nodal displacements

$$e_{xx}^{(m)} = \frac{\partial u_x^{(m)}}{\partial x} = \frac{U_{m+1} - U_m}{l_m}$$

$$= U_m \underset{\underset{J_m^{(m)}\left(\xi\right)}{\uparrow}}{\left(-\frac{1}{l_m}\right)} + U_{m+1} \underset{\underset{J_{m+1}^{(m)}\left(\xi\right)}{\uparrow}}{\left(\frac{1}{l_m}\right)}$$

shape functions for strain (constants in this case) again, if we let $J_j^{(m)} = 0 \quad j \neq m, m+1$ then we can write the strain in each element in terms of all the nodal variables

$$e_{xx}^{(m)} = \sum_{j=1}^{M+1} J_j^{(m)} U_j \quad \left(m = 1, \ldots, M \right)$$

3. Obtain virtual changes of the displacement and strain

$$\delta u_x^{(m)} = \sum_{j=1}^{M+1} H_j^{(m)} \delta U_j \quad \left(m = 1, \ldots, M \right)$$

$$\delta e_{xx}^{(m)} = \sum_{j=1}^{M+1} J_j^{(m)} \delta U_j \quad \left(m = 1, \ldots, M \right)$$

4. Apply Principle of Virtual Work

If $$\int_V f_x \delta u_x dV + \sum P_k \delta U_k = \int_V \sigma_{xx} \delta e_{xx} dV$$ is satisfied for all possible δU's then equilibrium will be (approximately) satisfied

work done by: distributed body force — concentrated loads

For concentrated loads, let $$d_j = \begin{cases} 1 \;\; if \;\; load \;\; at \;\; node \\ 0 \; otherwise \end{cases}$$ Then $$\sum P_k \delta U_k = \sum_{j=1}^{M+1} P_j d_j \delta U_j$$

and the principle gives

$$\sum_{m=1}^{M} \left\{ \int_{V_m} \underbrace{E \left[\sum_{k=1}^{M+1} J_k^{(m)} U_k \right]}_{\sigma_{xx}^{(m)} = E e_{xx}^{(m)}} \underbrace{\left[\sum_{j=1}^{M+1} J_j^{(m)} \delta U_j \right]}_{\delta e_{xx}^{(m)}} dV_m \right\} = \sum_{m=1}^{M} \left\{ \int_{V_m} f_x \underbrace{\left[\sum_{j=1}^{M+1} H_j^{(m)} \delta U_j \right]}_{\delta u_x^{(m)}} dV_m \right\} + \sum_{j=1}^{M+1} P_j d_j \delta U_j$$

$$\sum_{m=1}^{M}\left\{\int_{V_m} E\left[\sum_{k=1}^{M+1} J_k^{(m)}U_k\right]\left[\sum_{j=1}^{M+1} J_j^{(m)}\delta U_j\right]dV_m\right\} = \sum_{m=1}^{M}\left\{\int_{V_m} f_x\left[\sum_{j=1}^{M+1} H_j^{(m)}\delta U_j\right]dV_m\right\} + \sum_{j=1}^{M+1} P_j d_j \delta U_j$$

Rewrite as:

$$\sum_{j=1}^{M+1}\delta U_j\left\{\sum_{k=1}^{M+1}\left[\sum_{m=1}^{M}\int_{V_m} EJ_j^{(m)}J_k^{(m)}dV_m\right]U_k - \sum_{m=1}^{M}\left[\int_{V_m} f_xH_j^{(m)}dV_m\right] - P_jd_j\right\} = 0$$

For this to be true for all variations of δU_j

$$\sum_{k=1}^{M+1}\left[\sum_{m=1}^{M}\underbrace{\int_{V_m} EJ_j^{(m)}J_k^{(m)}dV_m}_{K_{jk}^{(m)}}\right]U_k = \underbrace{\sum_{m=1}^{M}\left[\int_{V_m} f_xH_j^{(m)}dV_m\right]}_{R_j} + P_jd_j \qquad (j=1,\ 2,\ \ldots, M+1)$$

$K_{jk}^{(m)}$
stiffness matrix
for each element

R_j
nodal forces produced
by distributed loads

$$K_{jk} = \sum_{m=1}^{M} K_{jk}^{(m)}$$ total stiffness matrix

$$\sum_{k=1}^{M+1}\left[\sum_{m=1}^{M}\int_{V_m} EJ_j^{(m)}J_k^{(m)}dV_m\right]U_k=\sum_{m=1}^{M}\left[\int_{V_m} f_x H_j^{(m)}dV_m\right]+P_j d_j \qquad (j=1,\, 2,\, \ldots,M+1)$$

⇨

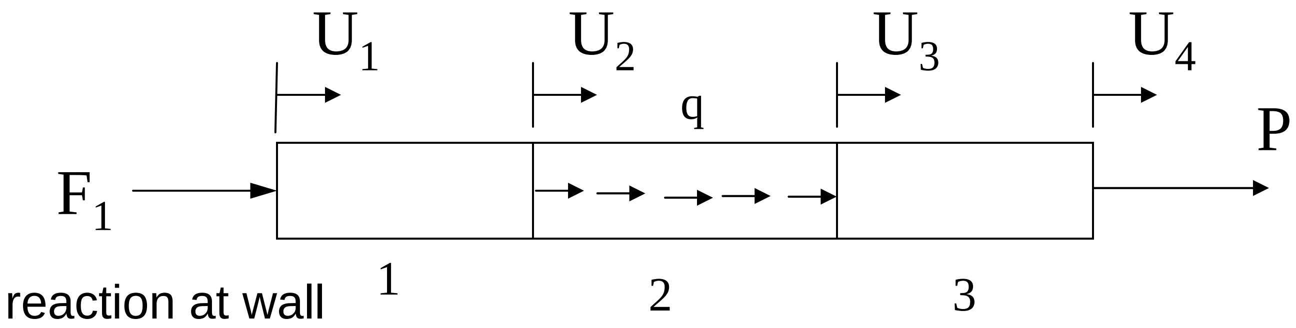
$$\sum_{k=1}^{M+1}K_{jk}U_k=R_j+P_j d_j$$

$$K_{jk}=\sum_{m=1}^{M}K_{jk}^{(m)}$$

$$R_j=\sum_{m=1}^{M}\left[\int_{V_m} f_x H_j^{(m)}dV_m\right]$$

$$K_{jk}^{(m)}=\sum_{m=1}^{M}\int_{V_m} EJ_j^{(m)}J_k^{(m)}dV_m$$

Now, consider our particular problem



$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} + \begin{Bmatrix} F_1 \\ 0 \\ 0 \\ P \end{Bmatrix}$$

6. Obtain stiffness matrix for each element:

$$K_{jk}^{(m)} = \sum_{m=1}^{M} \int_{V_m} E J_j^{(m)} J_k^{(m)} dV_m$$

$$= \sum_{m=1}^{3} EAl\, J_j^{(m)} J_k^{(m)}$$

$$J_m^{(m)}(\xi) = -\frac{1}{l}$$

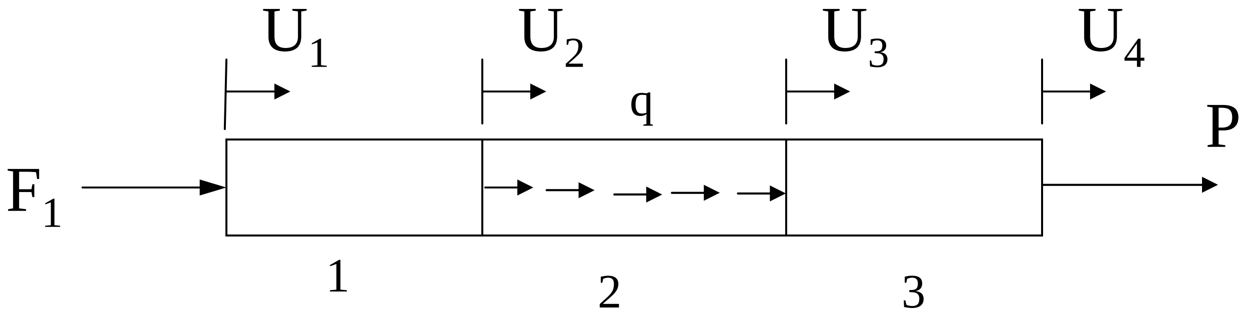
$$J_{m+1}^{(m)}(\xi) = \frac{1}{l}$$

all other $J_j^{(m)}(\xi) = 0$

$$K_{jk}^{(1)} : \begin{bmatrix} \frac{EA}{l} & \frac{-EA}{l} & 0 & 0 \\ \frac{-EA}{l} & \frac{EA}{l} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad K_{jk}^{(2)} : \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{EA}{l} & \frac{-EA}{l} & 0 \\ 0 & \frac{-EA}{l} & \frac{EA}{l} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad K_{jk}^{(3)} : \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{EA}{l} & \frac{-EA}{l} \\ 0 & 0 & \frac{-EA}{l} & \frac{EA}{l} \end{bmatrix}$$

7. Assemble total stiffness matrix:

$$K_{jk} = \sum_{m=1}^{3} K_{jk}^{(m)} : \begin{bmatrix} \frac{EA}{l} & \frac{-EA}{l} & 0 & 0 \\ \frac{-EA}{l} & \frac{2EA}{l} & \frac{-EA}{l} & 0 \\ 0 & \frac{-EA}{l} & \frac{2EA}{l} & \frac{-EA}{l} \\ 0 & 0 & \frac{-EA}{l} & \frac{EA}{l} \end{bmatrix}$$



7. Obtain all equivalent nodal forces from distributed loads:

$$R_j = \sum_{m=1}^{M} \left[\int_{V_m} f_x H_j^{(m)} dV_m \right]$$

$$= \int_{V_2} f_x H_j^{(2)} A d\xi$$

$$= q \int_0^l H_j^{(2)}(\xi) d\xi$$

$$H_2^{(2)}(\xi) = 1 - \frac{\xi}{l}$$

$$H_3^{(2)}(\xi) = \frac{\xi}{l}$$

all other $H_j^{(2)}(\xi) = 0$

$$\Rightarrow$$

$$R_1 = 0$$

$$R_2 = q\int_0^l \left(1 - \frac{\xi}{l}\right) d\xi = \frac{ql}{2}$$

$$R_3 = q\int_0^l \left(\frac{\xi}{l}\right) d\xi = \frac{ql}{2}$$

$$R_4 = 0$$

Resulting system cannot be solved directly because we have not eliminated rigid body motions

$$\begin{bmatrix} \frac{EA}{l} & \frac{-EA}{l} & 0 & 0 \\ \frac{-EA}{l} & \frac{2EA}{l} & \frac{-EA}{l} & 0 \\ 0 & \frac{-EA}{l} & \frac{2EA}{l} & \frac{-EA}{l} \\ 0 & 0 & \frac{-EA}{l} & \frac{EA}{l} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ ql/2 \\ ql/2 \\ 0 \end{Bmatrix} + \begin{Bmatrix} F_1 \\ 0 \\ 0 \\ P \end{Bmatrix}$$

8. Eliminate all possible rigid body motions and solve the system of equations

For this problem, if $U_j = 0$ set $K_{jj} = 1,\ K_{jk} = 0 \quad (k \neq j)$ and $R_j = P_j d_j = 0$

so that the *jth* equation gives $U_j = 0$ To preserve symmetry of the

stiffness matrix, also set $K_{kj} = 0$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{2EA}{l} & \frac{-EA}{l} & 0 \\ 0 & \frac{-EA}{l} & \frac{2EA}{l} & \frac{-EA}{l} \\ 0 & 0 & \frac{-EA}{l} & \frac{EA}{l} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ ql/2 \\ ql/2 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P \end{Bmatrix}$$