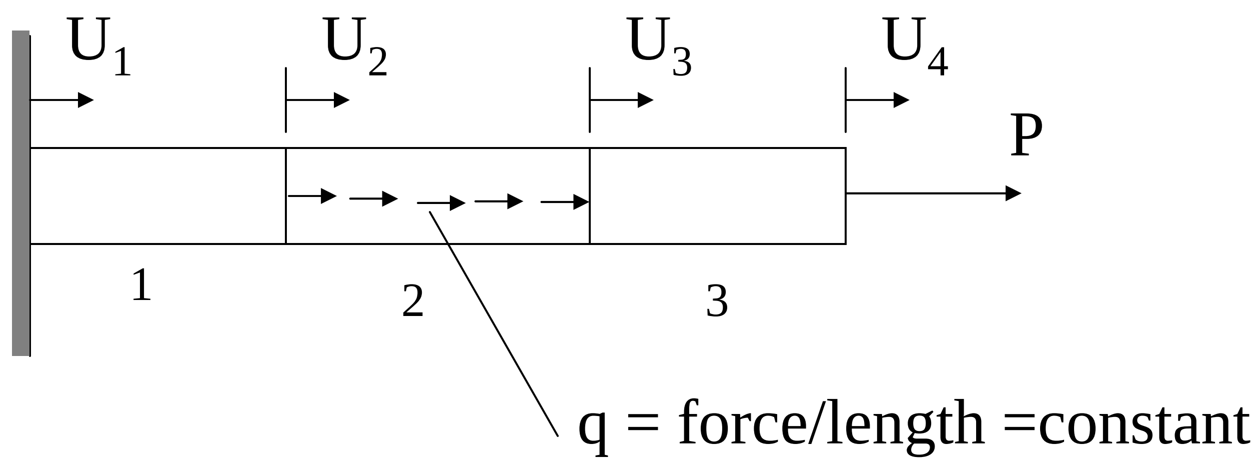
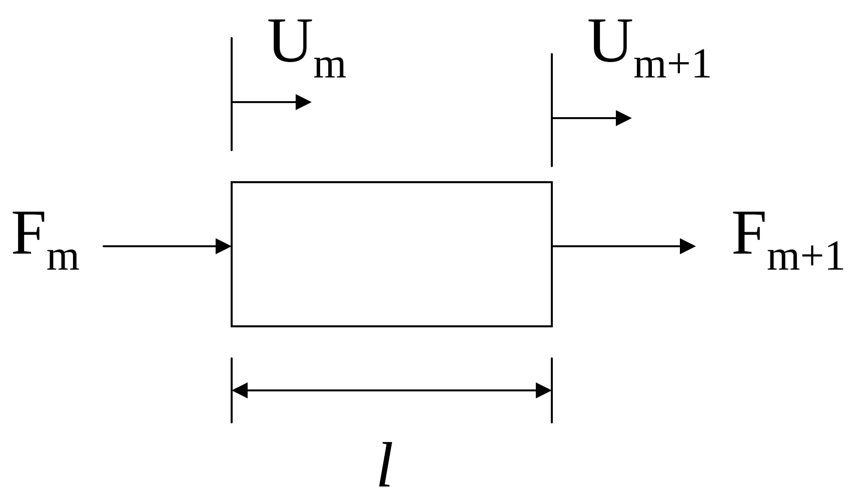
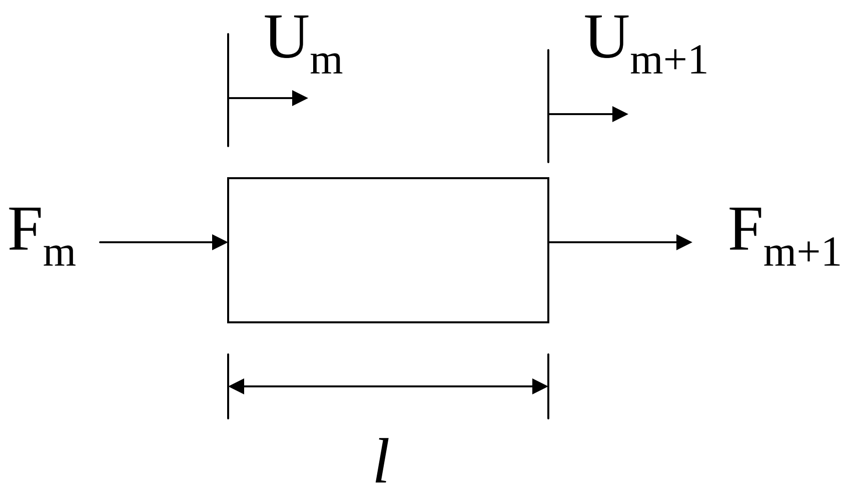
1-D Finite Elements - Equilibrium Approach





mth element



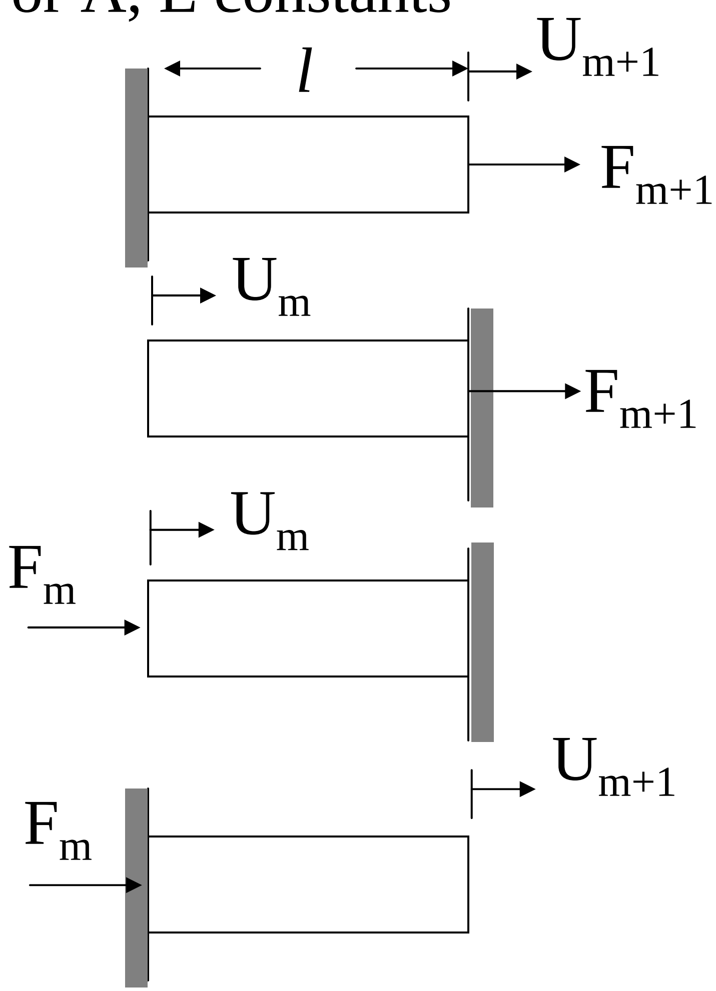
mth element

$$\begin{Bmatrix} F_{m+1} \\ F_m \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} U_{m+1} \\ U_m \end{Bmatrix}$$

K … stiffness matrix for the element

$$\begin{Bmatrix} F_{m+1} \\ F_m \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} U_{m+1} \\ U_m \end{Bmatrix}$$

For A, E constants



$F_{m+1} = K_{11}\, U_{m+1}$ if $U_m = 0$

$$K_{11} = \left.\frac{F_{m+1}}{U_{m+1}}\right|_{U_m=0} = \frac{AE}{l}$$

only true
for constant
strain element !

$$K_{12} = \left.\frac{F_{m+1}}{U_m}\right|_{U_{m+1}=0} = \frac{-AE}{l}$$

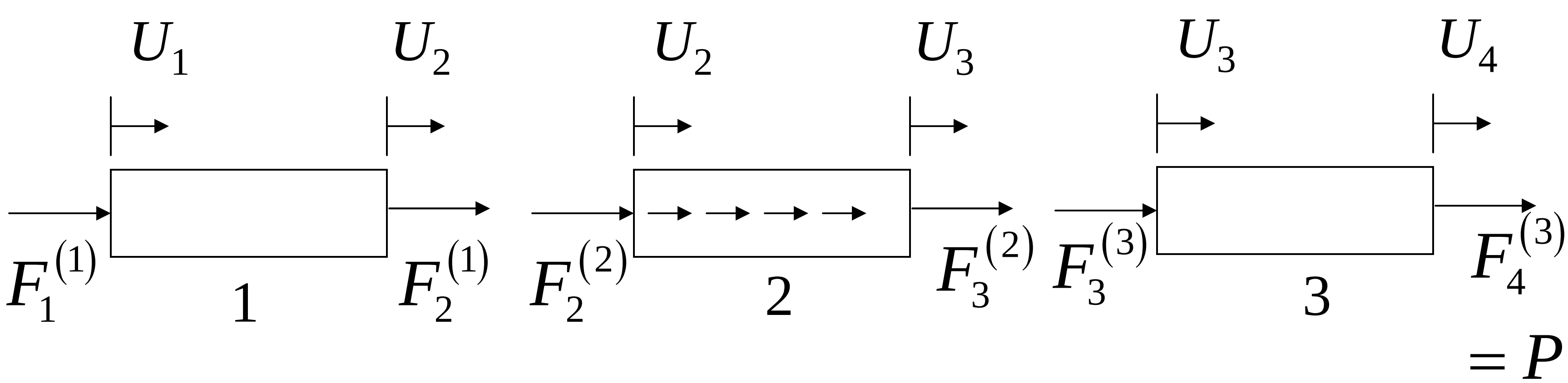
$$K_{22} = \left.\frac{F_m}{U_m}\right|_{U_{m+1}=0} = \frac{AE}{l}$$

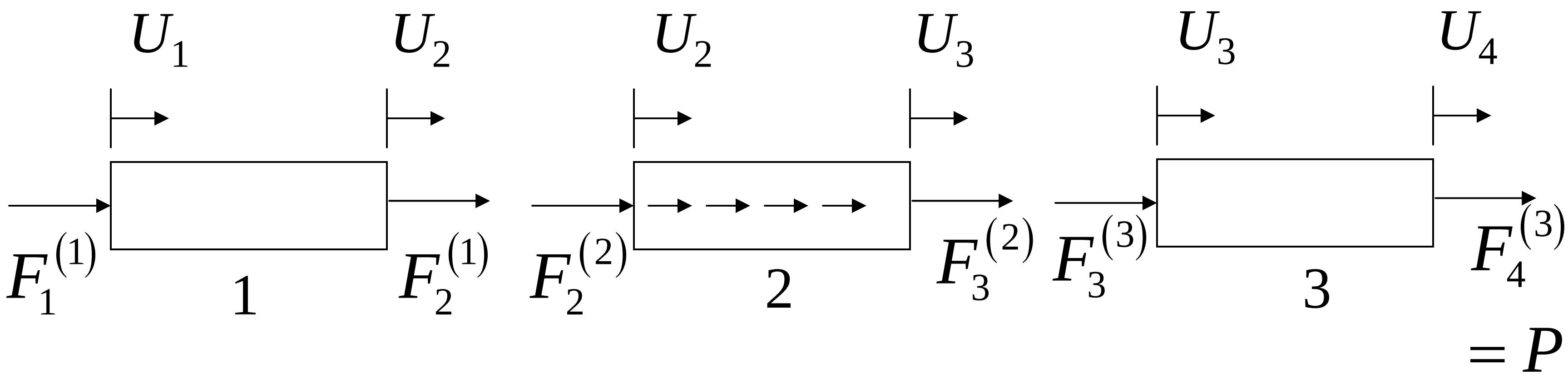
$$K_{21} = \left.\frac{F_m}{U_{m+1}}\right|_{U_m=0} = \frac{-AE}{l}$$

$$\begin{Bmatrix} F_{m+1} \\ F_m \end{Bmatrix} = \begin{bmatrix} \frac{AE}{l} & \frac{-AE}{l} \\ \frac{-AE}{l} & \frac{AE}{l} \end{bmatrix} \begin{Bmatrix} U_{m+1} \\ U_m \end{Bmatrix}$$

element stiffness matrix

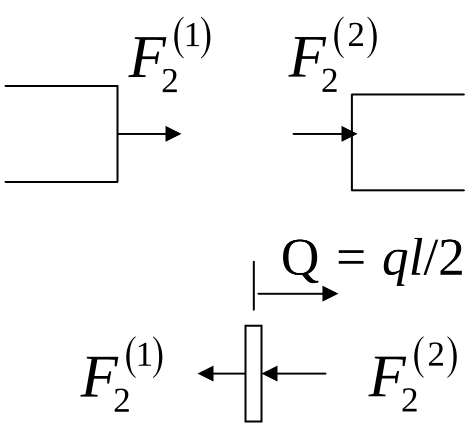
now, we need to assemble the elements





$$
\begin{bmatrix}
\frac{AE}{l} & \frac{-AE}{l} & 0 & 0 \\
\frac{-AE}{l} & \frac{AE}{l}+\frac{AE}{l} & \frac{-AE}{l} & 0 \\
0 & \frac{-AE}{l} & \frac{AE}{l}+\frac{AE}{l} & \frac{-AE}{l} \\
0 & 0 & \frac{-AE}{l} & \frac{AE}{l}
\end{bmatrix}
\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}
=
\begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)} + F_2^{(2)} \\ F_3^{(2)} + F_3^{(3)} \\ P \end{Bmatrix}
$$

$$
\begin{bmatrix}
\frac{AE}{l} & \frac{-AE}{l} & 0 & 0 \\
\frac{-AE}{l} & \frac{AE}{l}+\frac{AE}{l} & \frac{-AE}{l} & 0 \\
0 & \frac{-AE}{l} & \frac{AE}{l}+\frac{AE}{l} & \frac{-AE}{l} \\
0 & 0 & \frac{-AE}{l} & \frac{AE}{l}
\end{bmatrix}
\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}
=
\begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)} + F_2^{(2)} \\ F_3^{(2)} + F_3^{(3)} \\ P \end{Bmatrix}
$$



Place 1/2 of distributed
load at each end of the second element

$$\sum \mathbf{F} = 0$$

$$\Rightarrow \quad F_2^{(1)} + F_2^{(2)} = Q = ql/2$$

similarly $\quad F_3^{(2)} + F_3^{(3)} = ql/2$

apply B.C.
$U_1 = 0$

$$\begin{bmatrix} \frac{AE}{l} & \frac{-AE}{l} & 0 & 0 \\ \frac{-AE}{l} & \frac{2AE}{l} & \frac{-AE}{l} & 0 \\ 0 & \frac{-AE}{l} & \frac{2AE}{l} & \frac{-AE}{l} \\ 0 & 0 & \frac{-AE}{l} & \frac{AE}{l} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} F_1^{(1)} \\ ql/2 \\ ql/2 \\ P \end{Bmatrix}$$

Note: the full matrix is singular (determinant = 0) since an arbitrary constant displacement $U_1 = U_2 = U_3 = U_4 = C$ will satisfy the homogeneous equation. Another way to satisfy the boundary condition would be to set $K_{1m} = K_{m1} = 0 \quad (m \neq 1)$ and then solve the full 4x4 system.

$$K_{11} \neq 0$$

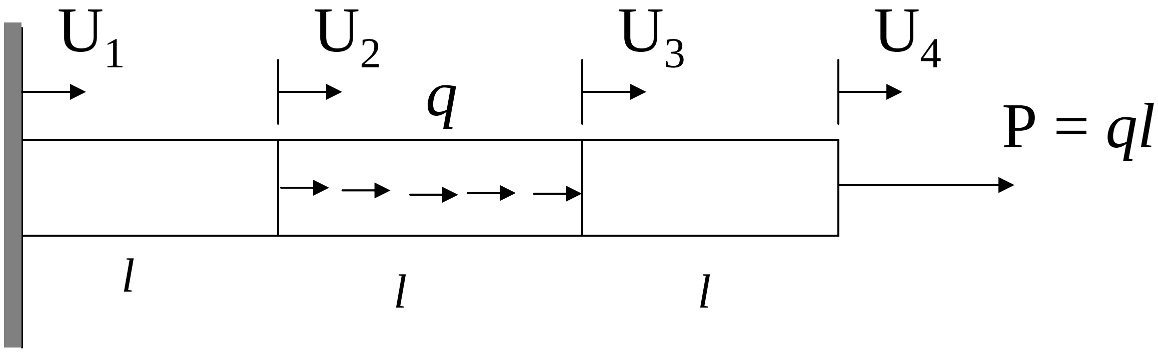
$$F_1^{(1)} = 0$$

$$\begin{bmatrix} \frac{2AE}{l} & \frac{-AE}{l} & 0 \\ \frac{-AE}{l} & \frac{2AE}{l} & \frac{-AE}{l} \\ 0 & \frac{-AE}{l} & \frac{AE}{l} \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} ql/2 \\ ql/2 \\ P \end{Bmatrix}$$

solving the 3x3 system yields the final solution

problem with the equilibrium approach - not simple to generalize for non-constant strains in the elements, varying q, varying A,E

Consider now the problem



$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} \tilde{U}_2 \\ \tilde{U}_3 \\ \tilde{U}_4 \end{Bmatrix} = \begin{Bmatrix} 1/2 \\ 1/2 \\ 1 \end{Bmatrix}$$

non-dimensional displacements $\tilde{U}_m = U_m \left(\dfrac{AE}{ql^2} \right)$

Finite Element Solution

```
» K=[ 2  -1  0;
      -1   2 -1;
       0  -1  1]
K =
   2   -1    0
  -1    2   -1
   0   -1    1

» F =[ 0.5  0.5  1]'

F =
  0.5000
  0.5000
  1.0000

» K\F
ans =

  2.0000
  3.5000
  4.5000
```

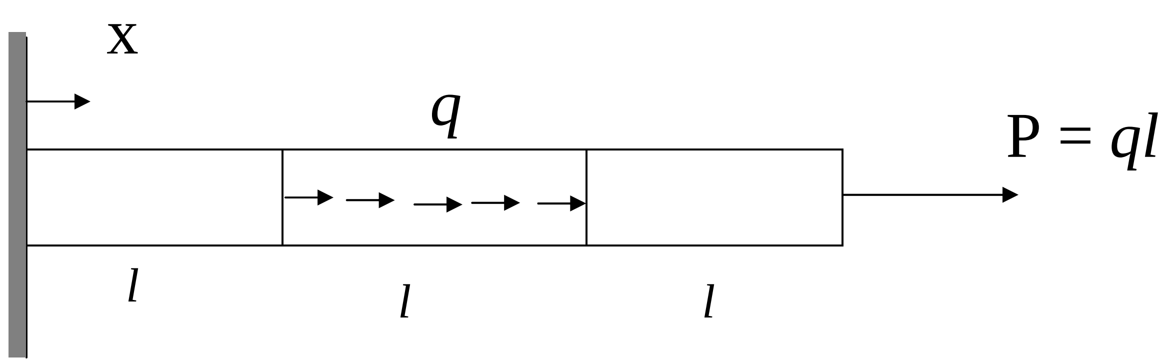
non-dimensional stiffness matrix, [K]

non-dimensional loads, {F}

solve [K]{U} = {F}

U_2
U_3
U_4

Exact Solution



$$u_x\left(\frac{AE}{ql^2}\right)=\begin{cases}2\left(\dfrac{x}{l}\right) & \left(0\le\dfrac{x}{l}\le 1\right)\\[2ex] 3\left(\dfrac{x}{l}\right)-\dfrac{1}{2}\left(\dfrac{x}{l}\right)^2-\dfrac{1}{2} & \left(1\le\dfrac{x}{l}\le 2\right)\\[2ex] \left(\dfrac{x}{l}\right)+\dfrac{3}{2} & \left(2\le\dfrac{x}{l}\le 3\right)\end{cases}$$

```
» % approximate (FEM) displacement in the bar
» x = linspace(0, 3, 100);
» ua =2*x.*(x<1) + (2 +1.5*(x-1)).*(x>=1 & x<=2) + (x+1.5).*(x>2);
»
» % exact displacement
» ue = 2*x.*(x<1) + (3*x -x.^2/2 - 0.5).*(x>=1 & x<=2) + (x+1.5).*(x>2);
» plot (x, ue, x, ua, '--')
» xlabel('x/l')
» ylabel('u*(AE/ql^2)')
```