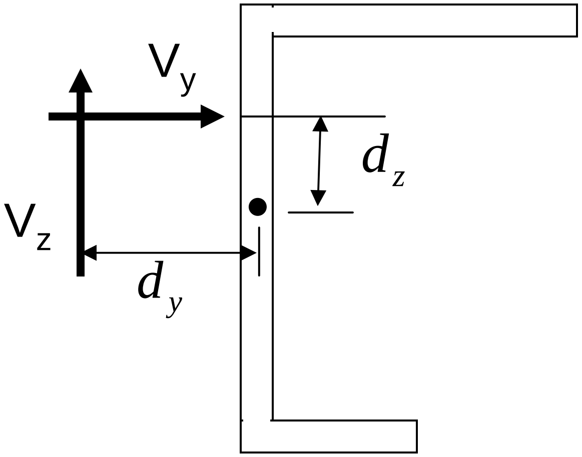
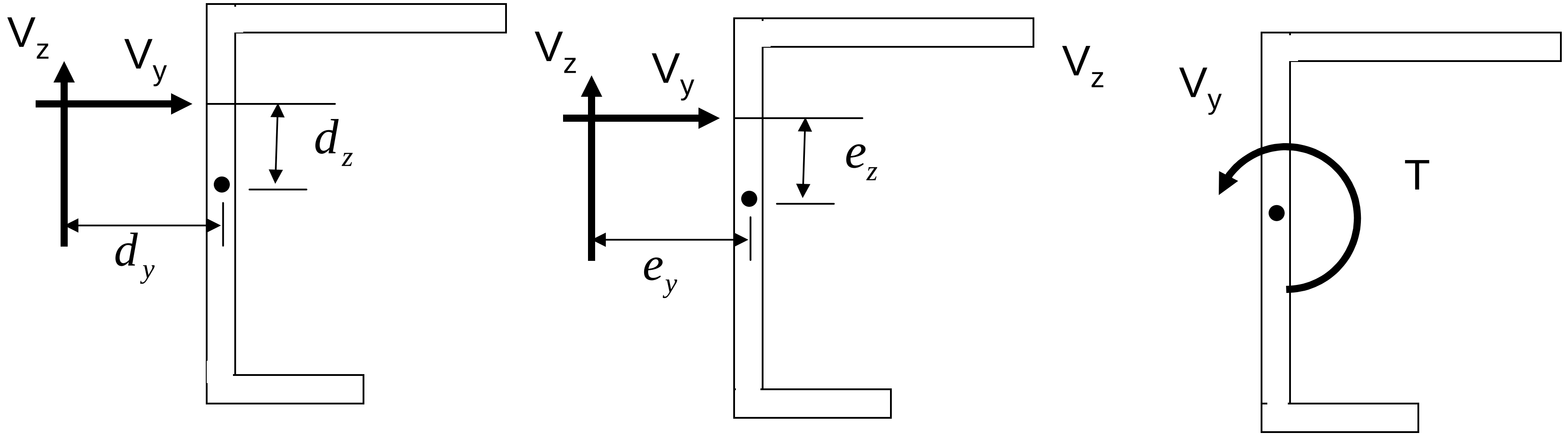
Bending and Torsion of Thin, Open Sections



If the shear forces and their locations are known, then we can calculate the shear flows assuming there is only bending produced, and from those shear flows (or through use of the sectorial area function) determine the location of the shear center.

Then the torsion induced by the shear forces can be calculated:

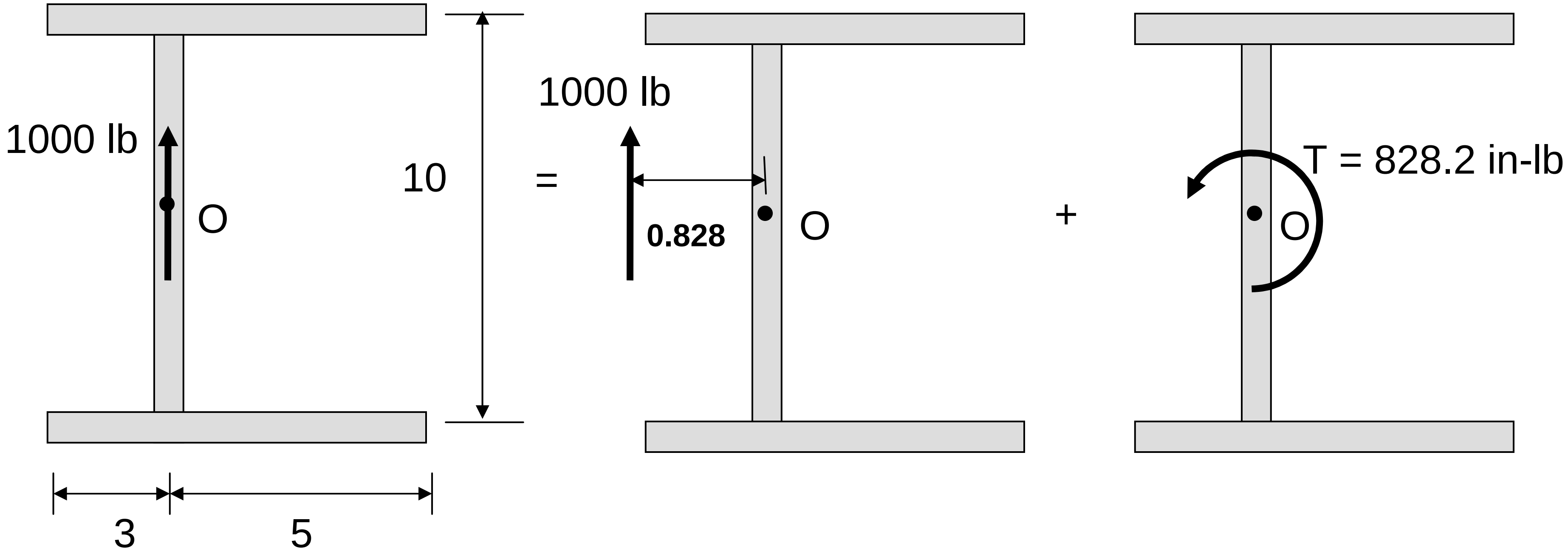


bending + torsion = bending only + torsion only

$$T = V_z\left(e_y - d_y\right) + V_y\left(e_z - d_z\right)$$

and we can solve for the shear stresses generated by T and superimpose them on the bending stresses to get the total shear stresses

As an example consider our previous problem but now let the 1000 lb load act through the vertical web:

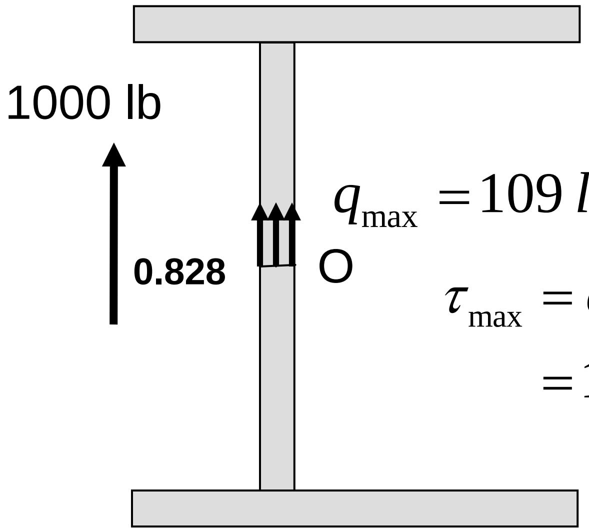


To determine the maximum shearing stress and its location (neglecting stress concentrations), for the torsional part we have

$$J_{eff} = 2\times\frac{1}{3}(8)(0.1)^3 + \frac{1}{3}(10)(0.1)^3$$
$$= 0.008666\, in^4$$

$$\tau_{\max} = \frac{Tt_{max}}{J_{eff}} = \frac{(828.2)(0.1)}{0.008666} = 9556\ psi$$

Now, the maximum shear stress will occur in the vertical web at the center where the shear flow is a maximum

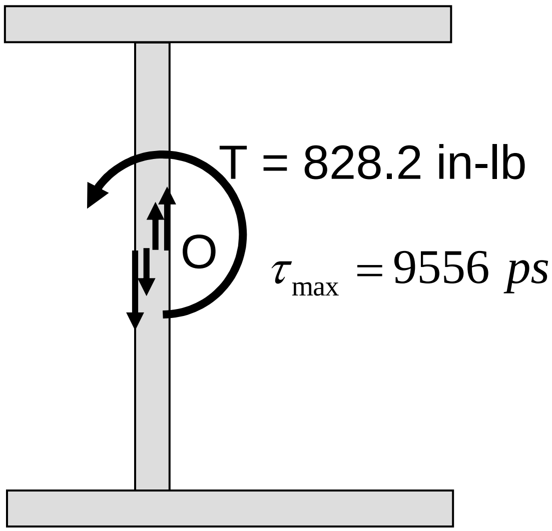


$$q_{\max} = 109\ lb/in$$

$$\tau_{\max} = q_{\max}/t$$

$$= 1090\ psi$$

bending

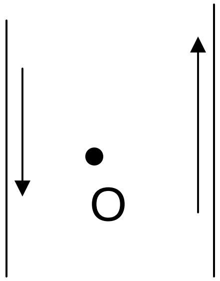


$$\tau_{\max} = 9556\ psi$$

torsion

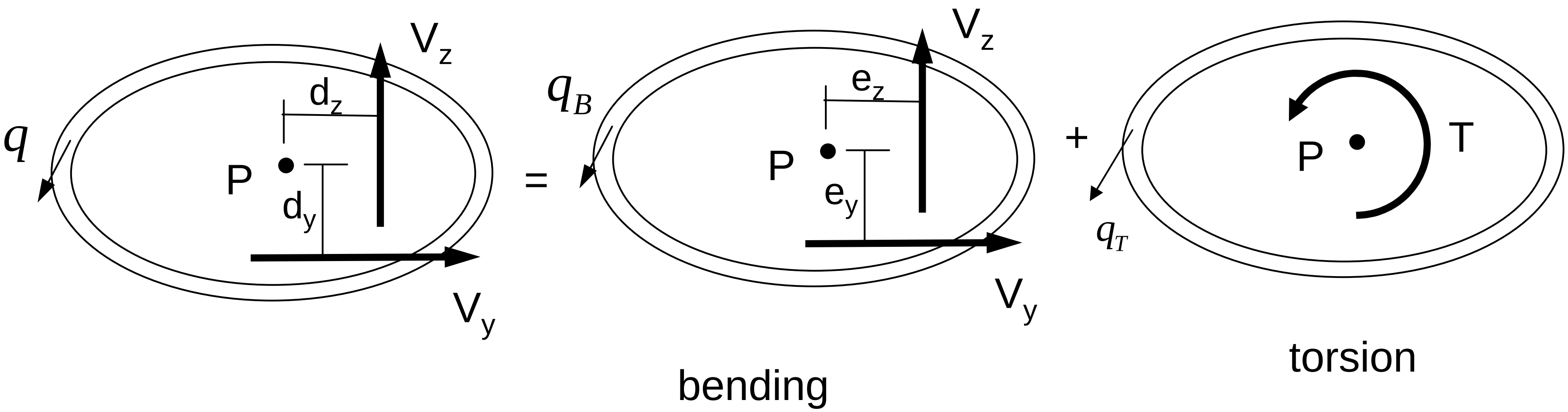
so at point O on the right side we will have

$$\tau = 8466\ psi$$



$$\tau_{\max} = 1090 + 9556 = 10,646\ psi$$

Bending and Torsion of a Thin Closed Section (single cell)



Note that the torsion that the shear forces generate will induce a constant shear flow q_T as shown. However in the bending induced, the shear flow is only known up to a constant since we have

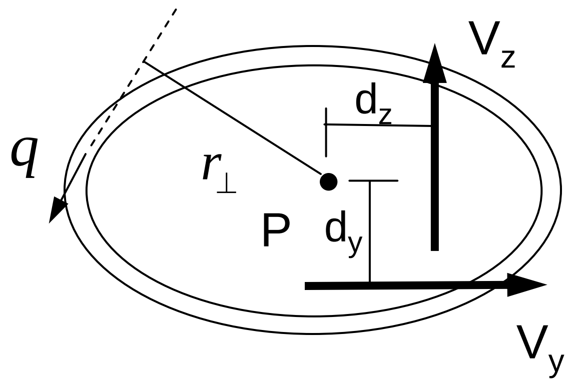
$$q_B\left(s\right)=q_B\left(0\right)+\frac{\left(V_zI_{yz}-V_yI_{yy}\right)Q_z+\left(V_yI_{yz}-V_zI_{zz}\right)Q_y}{D}$$

and, unlike open sections, we cannot find an end where $q_B = 0$

Thus, the total shear flow will be determined only to within a constant (which we must determine) that is partly due to bending and partly due to torsion.

Bending and Torsion of a Thin Closed Section (single cell)

Ω = area contained within the centerline of the cross section



$$\sum M_P = V_y d_y + V_z d_z = \oint_C q\, r_\perp ds$$

$$q(s) = \underset{\uparrow}{q_c} + V_y f(s) + V_z g(s)$$

this is the unknown constant shear flow due to both bending and torsion

$$f(s) = \frac{I_{yz} Q_y - I_{yy} Q_z}{I_{yy} I_{zz} - I_{yz}^2}$$

$$g(s) = \frac{I_{yz} Q_z - I_{zz} Q_y}{I_{yy} I_{zz} - I_{yz}^2}$$

Placing the q expression into the moment equation gives:

$$\sum M_P = 2\Omega\, q_c + \oint_C \left[V_y f(s) + V_z g(s) \right] r_\perp ds \qquad (1)$$

Also, recall we have a relationship between the shear flow and the twist/unit length induced by that shear flow

$$\phi' = \frac{1}{2G\Omega} \oint_C \frac{q}{t} ds \qquad (2)$$

1. If the shear forces and their positions are known, then q_c can be found directly from Eq.(1) since the left hand side of that equation is known explicitly and f and g can be found for the given geometry. Then Eq. (2) can be used (since q is now given completely) to find ϕ'

2. If the shear forces are known but assumed to act through the shear center (whose position is unknown), we can set $\phi' = 0$, $V_y = 0$ and solve Eq.(2) for the unknown q_c . Then Eq.(1) gives the location of the shear center, d_z , since

$$V_z d_z = \oint_C q \, r_\perp ds$$

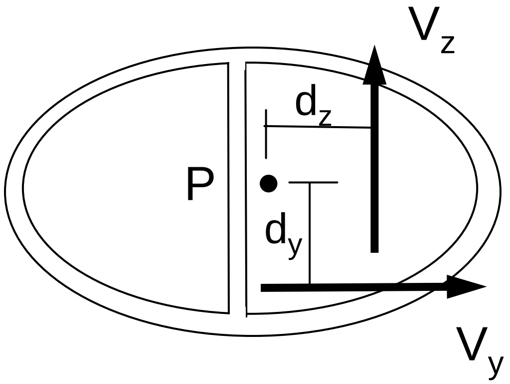
$$q(s) = q_c + V_z g(s)$$

We can repeat this process by setting $\phi' = 0$, $V_z = 0$ and solving Eq.(2) again for a new q_c . The Eq.(1) gives the location of the shear center, d_y :

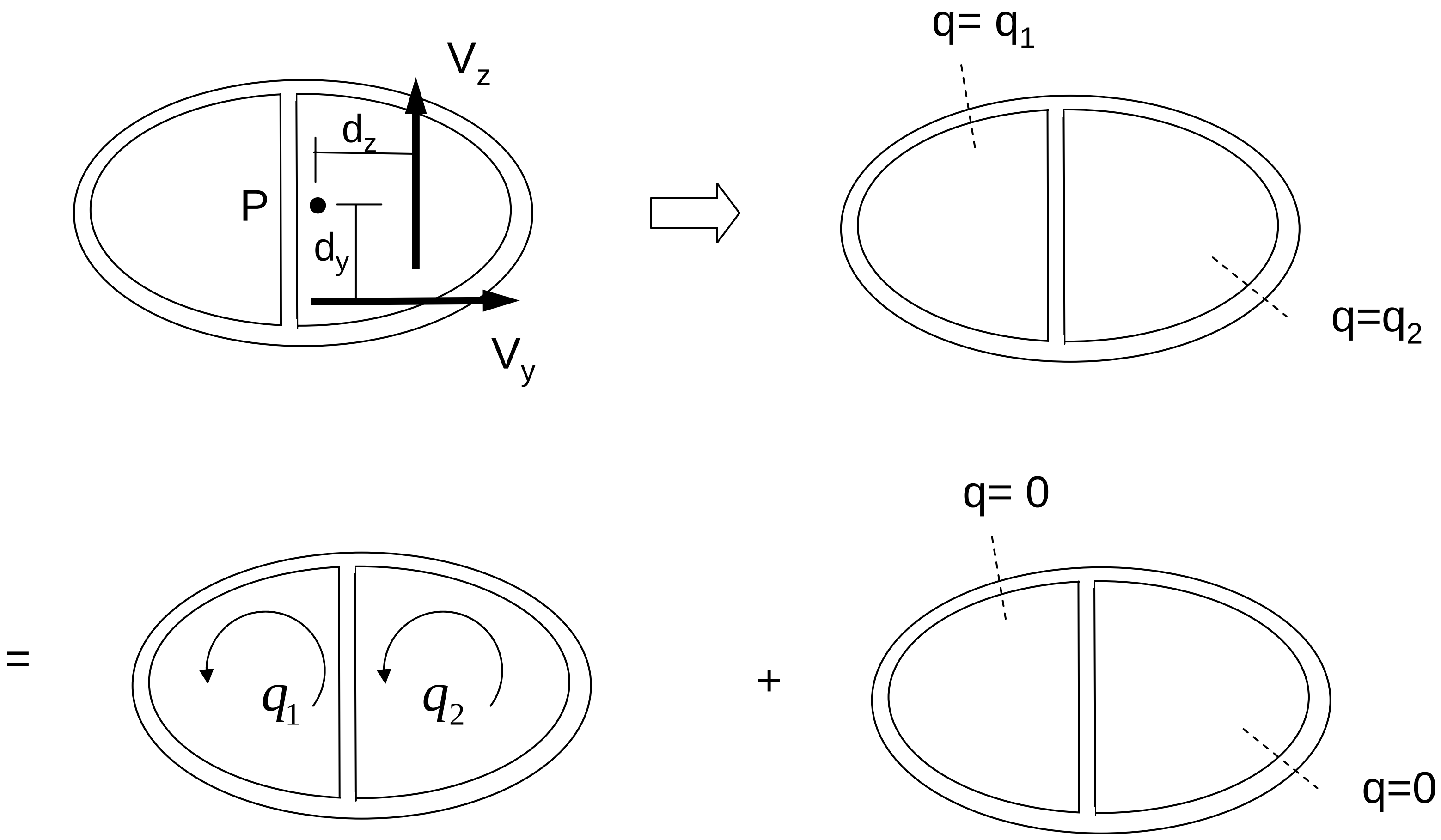
$$V_y d_y = \oint_C q \, r_\perp ds$$

$$q(s) = q_c + V_y f(s)$$

Bending and Torsion of a Thin Closed Section
(multiple cells)

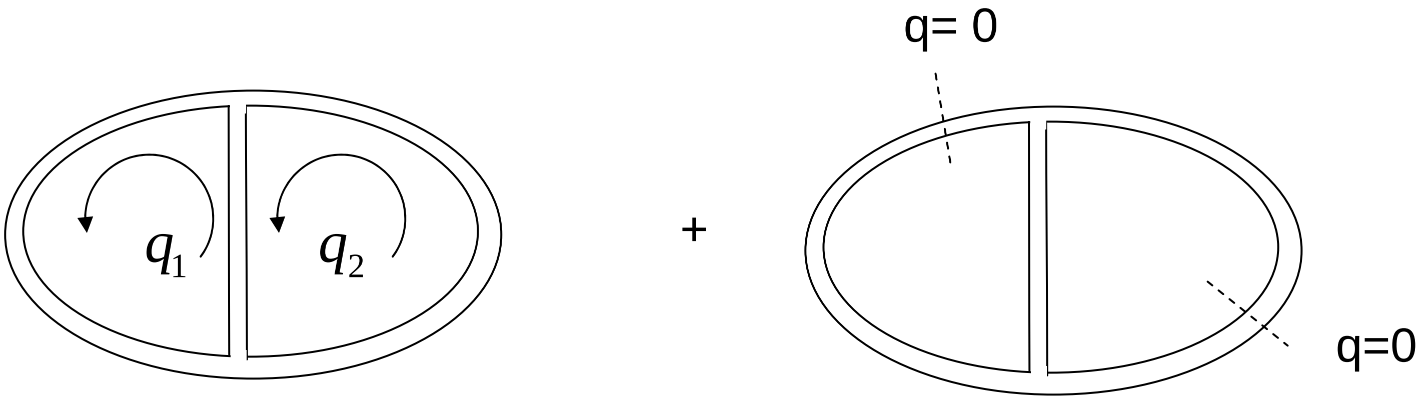


As in the case of the torsion of a multiple cell closed section, we need to account for unknown constant shear flows in each cell. This can be accomplished by conceptually decomposing our problem into two simpler problems as shown on the next page.



constant shear flows from constant parts of q due to bending and constant q's due to torsion.

bending shear flows q(s) due to this "open" section



$$V_y d_y + V_z d_z = 2\Omega_1 q_1 + 2\Omega_2 q_2 + \sum_{m=1}^{2} \oint_{C_m} \left[V_y f(s) + V_z g(s) \right] r_\perp ds \quad (1)$$

$$\phi' = \frac{1}{2G\Omega_1} \oint_{C_1} \frac{q}{t} ds \quad (2)$$

$$\phi' = \frac{1}{2G\Omega_2} \oint_{C_2} \frac{q}{t} ds \quad (3)$$

$$V_y d_y + V_z d_z = 2\Omega_1 q_1 + 2\Omega_2 q_2 + \sum_{m=1}^{2} \oint_{C_m} \left[V_y f(s) + V_z g(s) \right] r_\perp ds \quad (1)$$

$$\phi' = \frac{1}{2G\Omega_1} \oint_{C_1} \frac{q}{t} ds \quad (2)$$

$$\phi' = \frac{1}{2G\Omega_2} \oint_{C_2} \frac{q}{t} ds \quad (3)$$

1. If the shear forces and their locations are known, then q_1 and q_2 are first found in terms of the unknown ϕ' from Eqs. (2) and (3). These q_m 's are then placed into Eq.(1) which is solved for the unknown ϕ' . Once ϕ' is known in this manner, the q_m 's are completely determined.

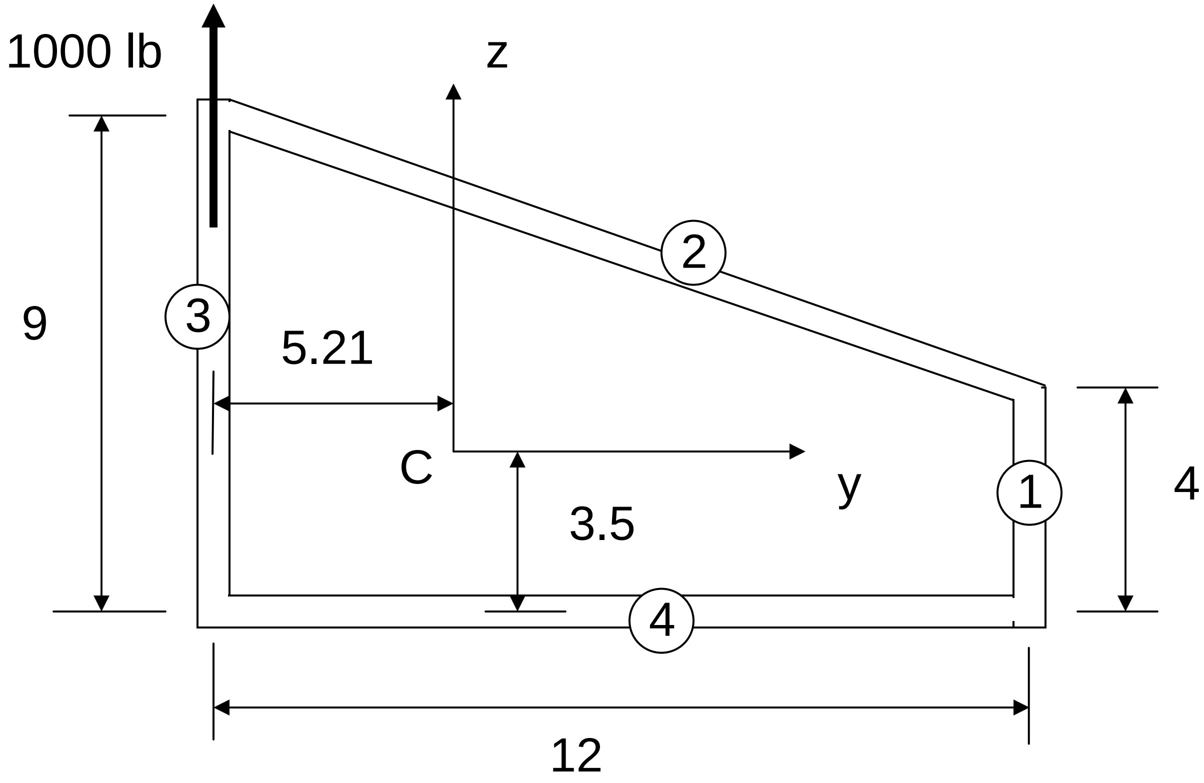
2. If the shear forces are known but assumed to act through the shear center (whose position is unknown), we can set ϕ' =0, V_y = 0 and solve Eqs. (2) and (3) for the unknowns q_1 and q_2 . Then Eq.(1) gives the location of the shear center, d_z , since

$$V_z d_z = 2\Omega_1 q_1 + 2\Omega_2 q_2 + \sum_{m=1}^{2} \oint_{C_m} \left[V_z g(s) \right] r_\perp ds$$

We can repeat this process by setting $\phi' = 0$, $V_z = 0$ and solving Eqs.(2) and (3) again for new values q_1 and q_2 . Then Eq.(1) gives the location of the shear center, d_y :

$$V_y d_y = 2\Omega_1 q_1 + 2\Omega_2 q_2 + \sum_{m=1}^{2} \oint_{C_m} \left[V_y f\left(s\right) \right] r_{\perp} ds$$

Example; Calculate the shear flows in the walls of the box beam shown. The beam has a uniform wall thickness of t = 0.1 in. all dimensions are in inches



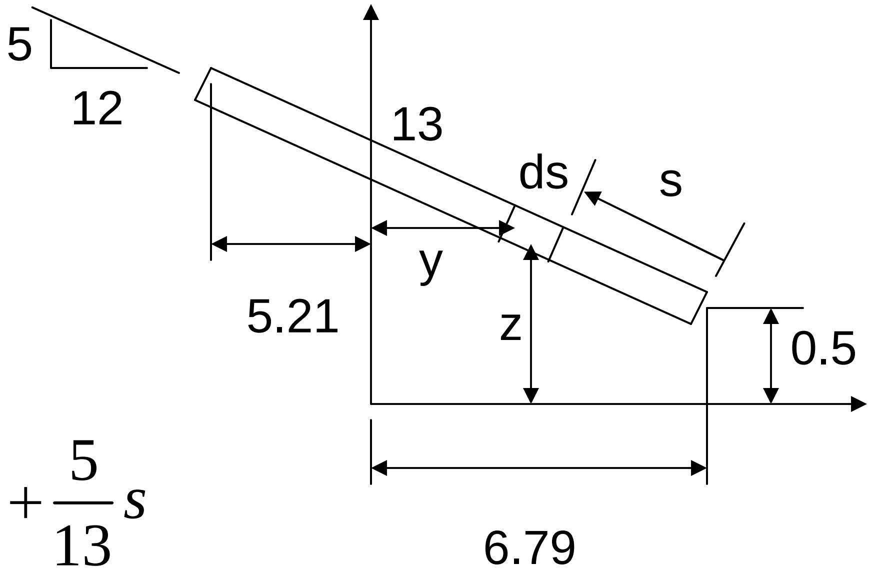
$$I_{yy}^{(1)} = 1.433 \quad I_{zz}^{(1)} = 18.44 \quad I_{yz}^{(1)} = -4.074$$

$$I_{yy}^{(2)} = ? \quad I_{zz}^{(2)} = ? \quad I_{yz}^{(2)} = ?$$

$$I_{yy}^{(3)} = 6.975 \quad I_{zz}^{(3)} = 24.43 \quad I_{yz}^{(3)} = -4.689$$

$$I_{yy}^{(4)} = 14.70 \quad I_{zz}^{(4)} = 15.15 \quad I_{yz}^{(1)} = -3.318$$

To find the I's for ②



$$z = 0.5 + \frac{5}{13}s$$

$$y = 6.79 - \frac{12}{13}s$$

$$I_{yy}^{(2)} = \int_0^{13} z^2 dA$$

$$= \int_0^{13} \left(0.5 + 5s/13\right)^2 \left(0.1\right) ds$$

$$= 14.41\, in^4$$

similarly $$I_{zz}^{(2)} = \int_0^{13} y^2 dA = 16.41\, in^4$$

$$I_{yz}^{(2)} = \int_0^{13} yz dA = -3.419\, in^4$$

so total area moments are

$$I_{yy} = \sum_{m=1}^{4} I_{yy}^{(m)} = 37.52\, in^4$$

$$I_{zz} = \sum_{m=1}^{4} I_{zz}^{(m)} = 74.43\, in^4$$

$$I_{yz} = \sum_{m=1}^{4} I_{yz}^{(m)} = -15.50\, in^4$$

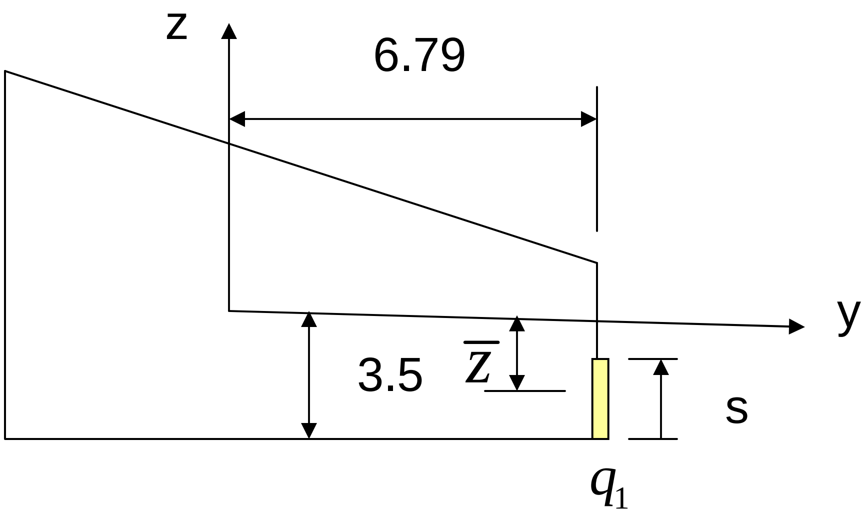
$$V_z = 1000\, lb, \quad V_y = 0$$

$$\Delta q = \frac{-\left(I_{zz}Q_y - I_{yz}Q_z\right)V_z}{I_{yy}I_{zz} - I_{yz}^2}$$

$$= \frac{\left(-74.43Q_y - 15.5Q_z\right)\left(1000\right)}{\left(37.52\right)\left(74.43\right) - \left(15.5\right)^2}$$

⇨ $$\Delta q = -29.16\, Q_y - 6.073\, Q_z$$

Now, use this relation for each section



$$Q_y^{(1)} = \int z\,dA = -\underset{\bar{z}}{\left(3.5 - s/2\right)}\underset{A}{\left(0.1\right)\left(s\right)}$$

$$= -0.35s + 0.05s^2$$

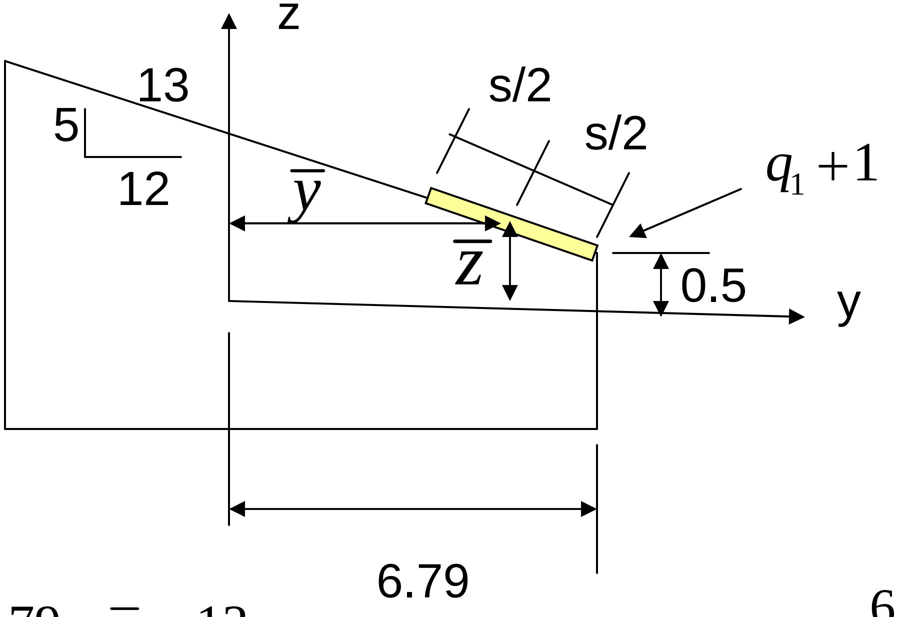
$$Q_z^{(1)} = \int y\,dA = \underset{\bar{y}}{\left(6.79\right)}\underset{A}{\left(0.1\right)\left(s\right)}$$

$$= 0.679s$$

$$q^{(1)}\left(s\right) = q_1 - 29.16\left[-0.35s + 0.05s^2\right] - 6.073\left[0.679s\right]$$

$$= q_1 + 6.082s - 1.458s^2 \quad lb/in$$

$$q^{(1)}\left(4\right) = q_1 + 1 \;\; lb/in$$



$$\frac{6.79-\overline{y}}{s/2}=\frac{12}{13} \qquad \Rightarrow \qquad \overline{y}=6.79-\frac{6}{13}s$$

$$\frac{\overline{z}-0.5}{s/2}=\frac{5}{13} \qquad \qquad \overline{z}=0.5+\frac{5}{26}s$$

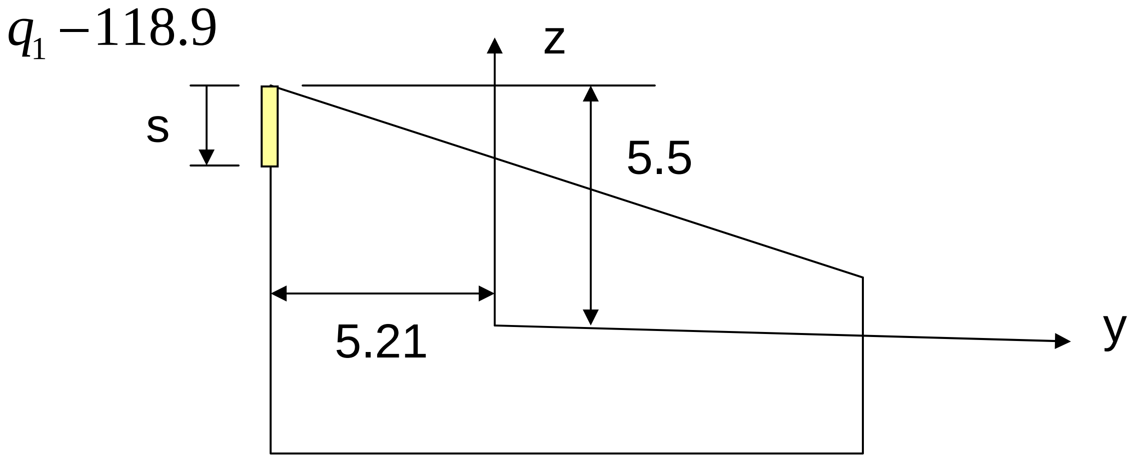
$$Q_y^{(2)}=\left(0.5+5s/26\right)\left(0.1\right)\left(s\right)=0.05s+0.1923s^2$$

$$Q_z^{(2)}=\left(6.79-6s/13\right)\left(0.1\right)\left(s\right)=0.679s-0.04615s^2$$

$$q^{(2)}\left(s\right)=q_1+1-29.16\left[0.05s+0.01923s^2\right]-6.073\left[0.679s-0.04615s^2\right]$$

$$q^{(2)}\left(s\right)=q_1+1-5.58s-0.2802s^2 \quad lb/in$$

$$q^{(2)}\left(13\right)=q_1-118.9 \ \ lb/in$$



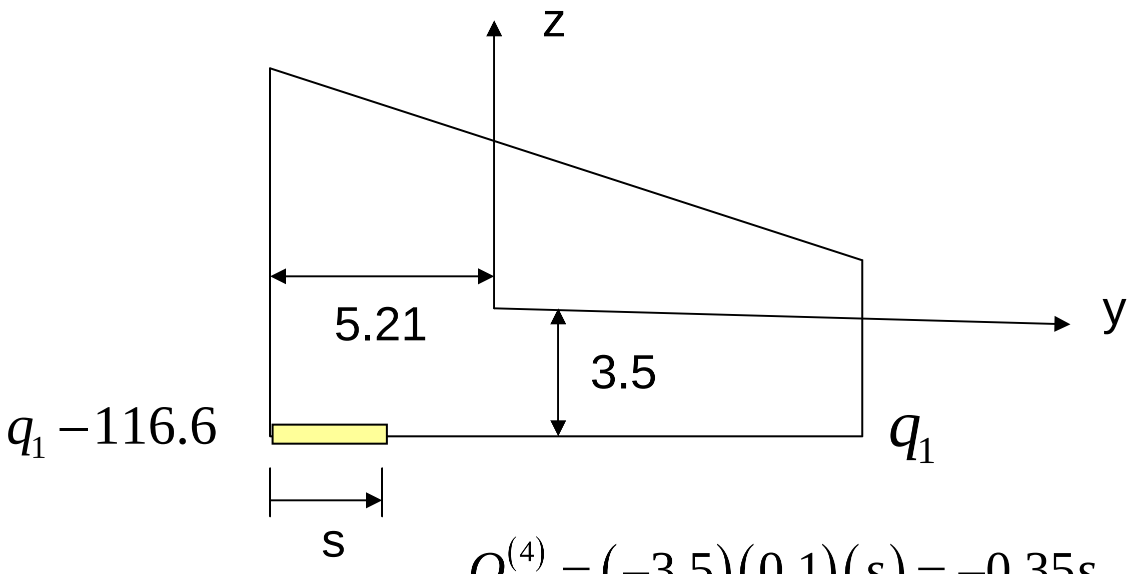
$$Q_y^{(3)} = \left(5.5 - s/2\right)\left(0.1\right)\left(s\right) = 0.55s - 0.05s^2$$

$$Q_z^{(3)} = \left(-5.21\right)\left(0.1\right)\left(s\right) = -0.521s$$

$$q^{(3)}\left(s\right) = q_1 - 118.9 - 29.16\left[0.55s - 0.05s^2\right] - 6.073\left[-0.521s\right]$$

$$q^{(3)}\left(s\right) = q_1 - 118.9 - 12.87s + 1.458s^2 \quad lb/in$$

$$q^{(3)}\left(9\right) = q_1 - 116.6 \;\; lb/in$$



$$Q_y^{(4)} = (-3.5)(0.1)(s) = -0.35s$$

$$Q_z^{(4)} = -(5.21 - s/2)(0.1)(s) = -0.521s + 0.05s^2$$

$$q^{(4)}(s) = q_1 - 116.6 - 29.16[-0.35s] - 6.073\left[-0.521s + 0.05s^2\right]$$

$$q^{(4)}(s) = q_1 - 116.6 + 13.37s - 0.3036s^2 \quad lb/in$$

$$q^{(4)}(12) = q_1 - 116.6 + 160.44 - 43.72 = q_1$$ (approximately)

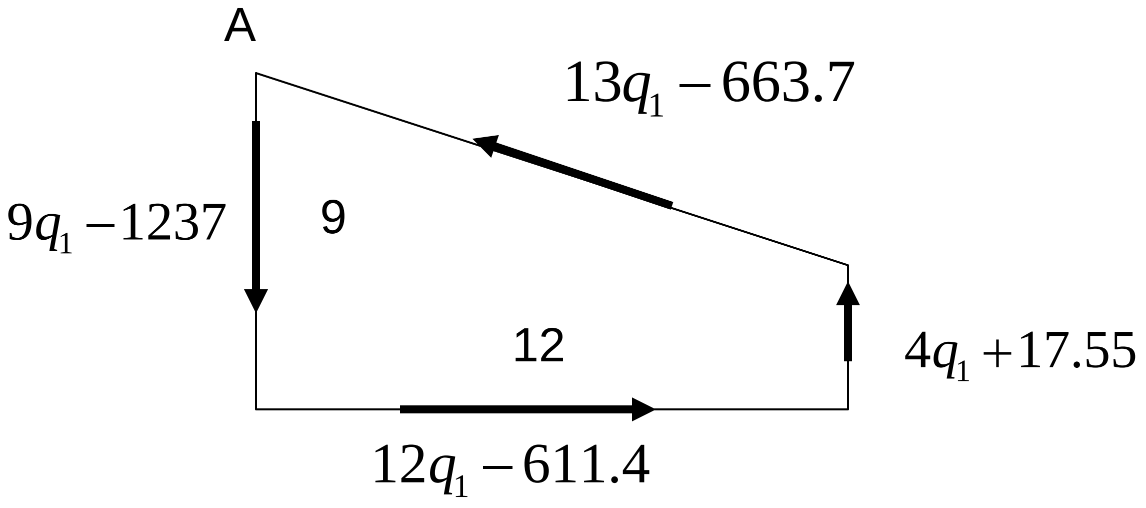
total force in each wall

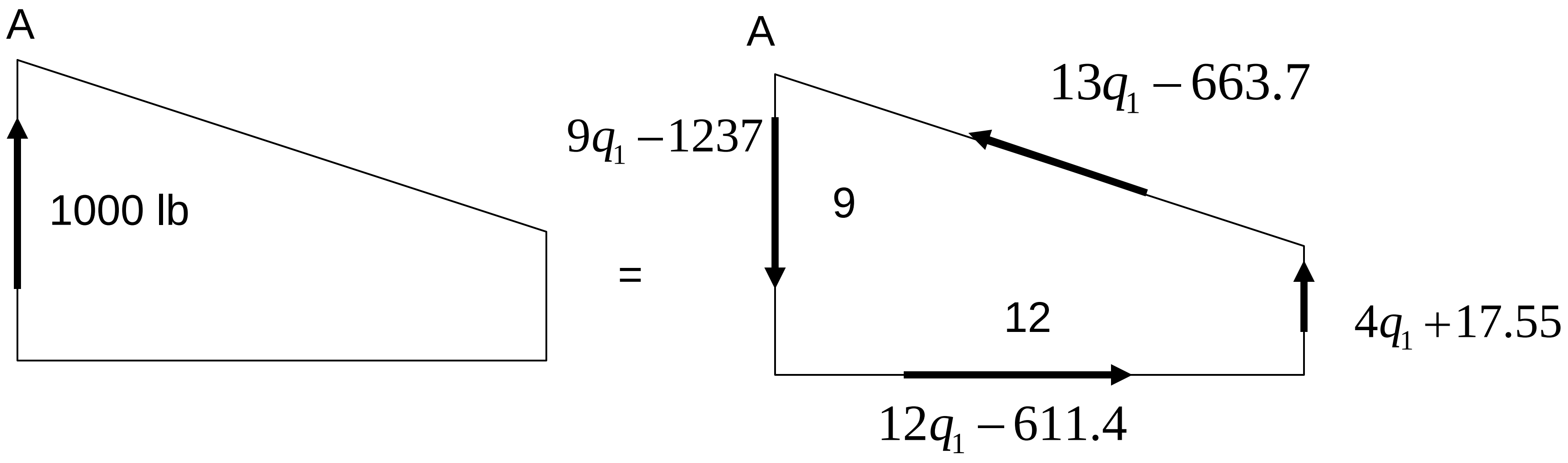
$$F^{(1)} = \int_0^4 \left(q_1 + 6.082s - 1.458s^2 \right) ds = 4q_1 + 17.55 \quad lb$$

$$F^{(2)} = \int_0^4 \left(q_1 + 1 - 5.58s - 0.2802s^2 \right) ds = 13q_1 - 663.7 \quad lb$$

$$F^{(3)} = \int_0^4 \left(q_1 - 118.9 - 12.87s + 1.458s^2 \right) ds = 9q_1 - 1237 \quad lb$$

$$F^{(4)} = \int_0^4 \left(q_1 - 116.6 + 13.37s - 0.3036s^2 \right) ds = 12q_1 + 611.4 \quad lb$$





to find q_1 we must equate moment about any point generated by the shear flows to the moment due to the applied load

$$\sum M_A = 0$$

$$\left(2q_1 - 116.4\right)\left(9\right) + \left(4q_1 + 17.55\right)\left(12\right) = 0$$

$$q_1 = 33.92\ lb/in$$

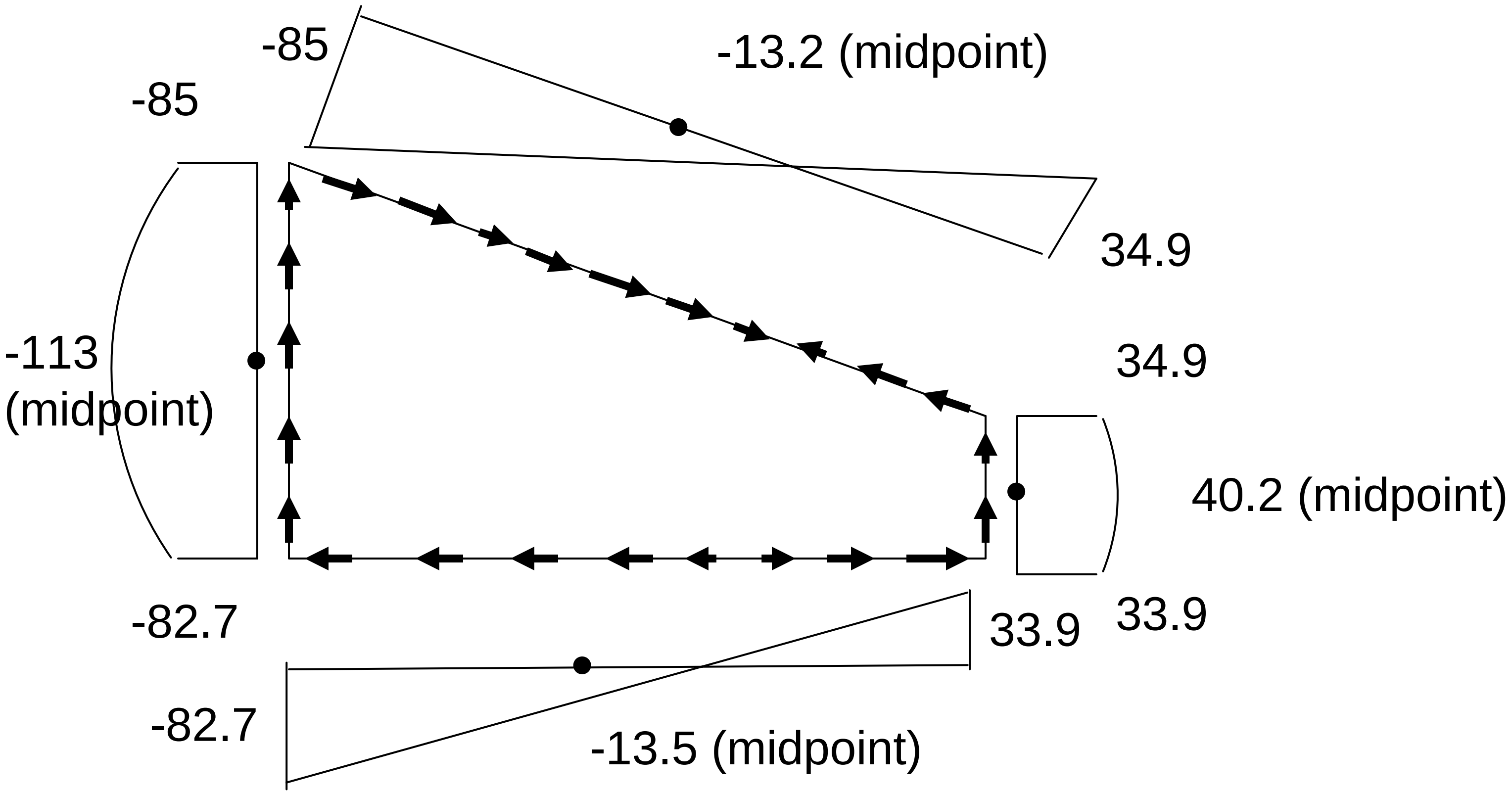
then the shear flows are explicitly

$$q^{(1)}(s) = 33.9 + 6.08s - 1.46s^2 \quad lb/in$$

$$q^{(2)}(s) = 34.9 - 5.58s - 0.208s^2 \quad lb/in$$

$$q^{(3)}(s) = -85.0 - 12.9s + 1.46s^2 \quad lb/in$$

$$q^{(4)}(s) = -82.7 + 13.4s - 0.304s^2 \quad lb/in$$



Now, find the location of the shear center for the vertical load

We need to set $\phi' = \dfrac{1}{2G\Omega} \oint \dfrac{q}{t} ds = 0$

and find the q_1 that meets this requirement

Since t is a constant here we have equivalently

$$\oint q ds = 0$$

which is just the sum of the forces we already computed

$$\left(4q_1 + 17.55\right) + \left(13q_1 - 663.7\right) + \left(9q_1 - 1237\right) + \left(2q_1 - 611.4\right) = 0$$

which gives

$$q_1 = 65.65 \;\; lb/in$$

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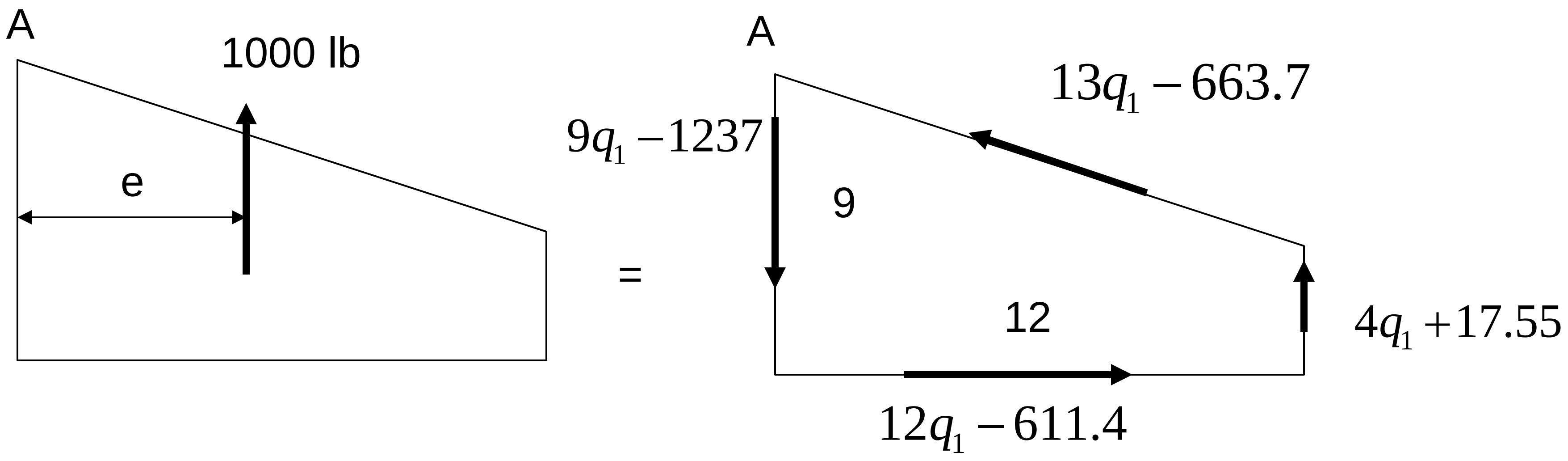
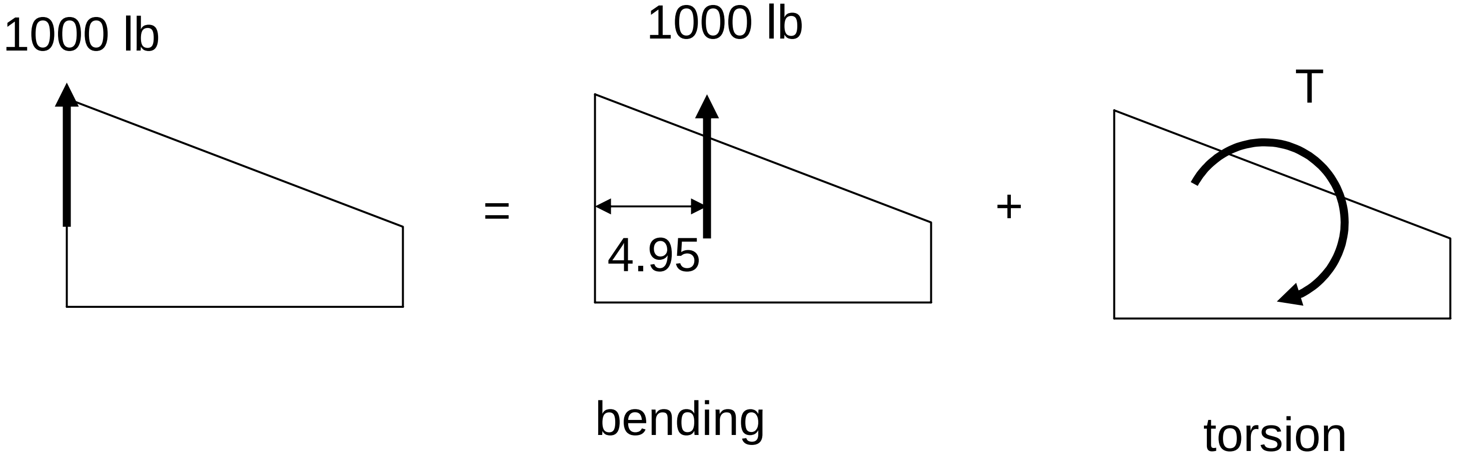


$$\sum M_A = 1000e$$

$$\left(4q_1 + 17.55\right)\left(12\right) + \left(12q_1 - 611.4\right)\left(9\right) = 1000e$$

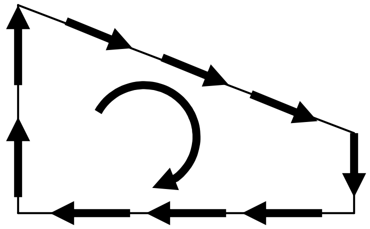
$$e = 4.95\,in$$

If we want to find the vertical location of the shear center we would have to solve this problem for a horizontal applied shear force



shear flow due to torsion (constant around the whole cross section)

$$q = \frac{T}{2\Omega} = \frac{4950}{2\left[(4)(12)+(5)(12)/2\right]} = 31.7 \ \ lb/in$$



Thus, the q_1 in our original problem is due partially to the bending part of the solution, partially due to torsion