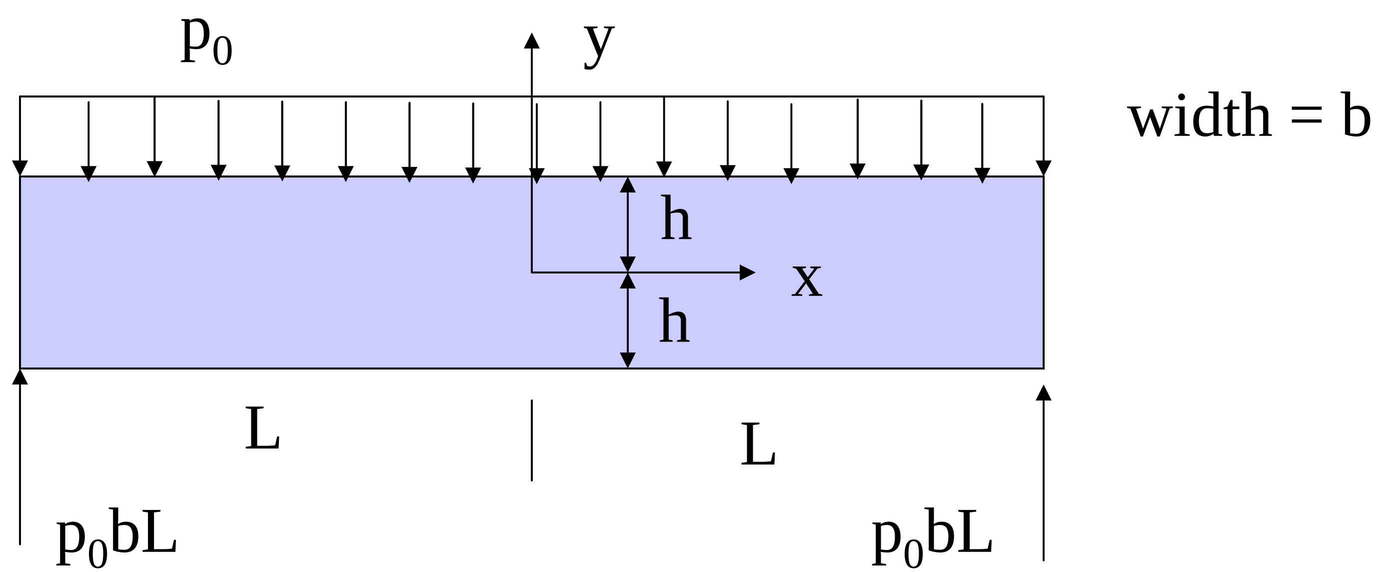


p0
y
width = b
h
x
h
L
L
p0bL
p0bL

Try

$$\sigma_{xx} = c_1 y + c_2 x^2 y + c_3 y^3$$

$$\sigma_{xy} = c_4 x + c_5 x y^2$$

$$\sigma_{yy} = c_6 + c_7 y + c_8 y^3$$

To satisfy equilibrium:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

$\Rightarrow$

$$c_5 = -c_2$$

$$c_4 = -c_7$$

$$c_5 = -3c_8$$

$$\sigma_{xx} = c_1 y + 3c_8 x^2 y + c_3 y^3$$
$$\sigma_{xy} = c_4 x - 3c_8 x y^2$$
$$\sigma_{yy} = c_6 - c_4 y + c_8 y^3$$

To satisfy compatibility

$$\nabla^2 \left(\sigma_{xx} + \sigma_{yy} \right) = 0 \quad \Rightarrow \quad 12c_8 + 6c_3 = 0$$

$$\sigma_{xx} = c_1 y + 3c_8 x^2 y - 2c_8 y^3$$
$$\sigma_{xy} = c_4 x - 3c_8 x y^2$$
$$\sigma_{yy} = c_6 - c_4 y + c_8 y^3$$

To make them simpler, write the stresses as

$$\sigma_{xx} = ay + 3bx^2 y - 2by^3$$

$$\sigma_{xy} = cx - 3bxy^2$$

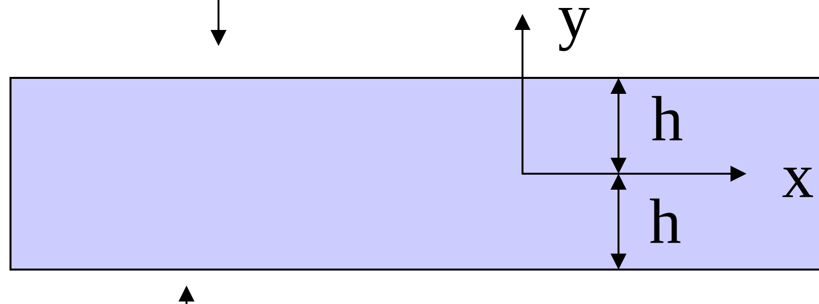
$$\sigma_{yy} = d - cy + by^3$$

Now, we need to apply the boundary conditions to solve for *a*, *b*, *c*, *d*

Boundary Conditions

$$\sigma_{yy}\left(x,h\right) = -p_0 \qquad \text{(I)}$$

$$\sigma_{xy}\left(x,h\right) = 0 \qquad \text{(II)}$$



$$\sigma_{yy}\left(x,-h\right) = 0 \qquad \text{(III)}$$

$$\sigma_{xy}\left(x,-h\right) = 0 \qquad \text{(IV)}$$

(II, IV): At $y = \pm h$ $\sigma_{xy} = 0$ $\Rightarrow$ $c = 3bh^2$

(I, III):

At $y = h$ $\sigma_{yy} = -p_0$ $\Rightarrow$ $-p_0 = d - 2bh^3$

$y = -h$ $\sigma_{yy} = 0$ $\qquad 0 = d + 2bh^3$

so $d = -p_0 / 2$, $b = p_0 / 4h^3$ and $c = 3p_0 / 4h$

which gives

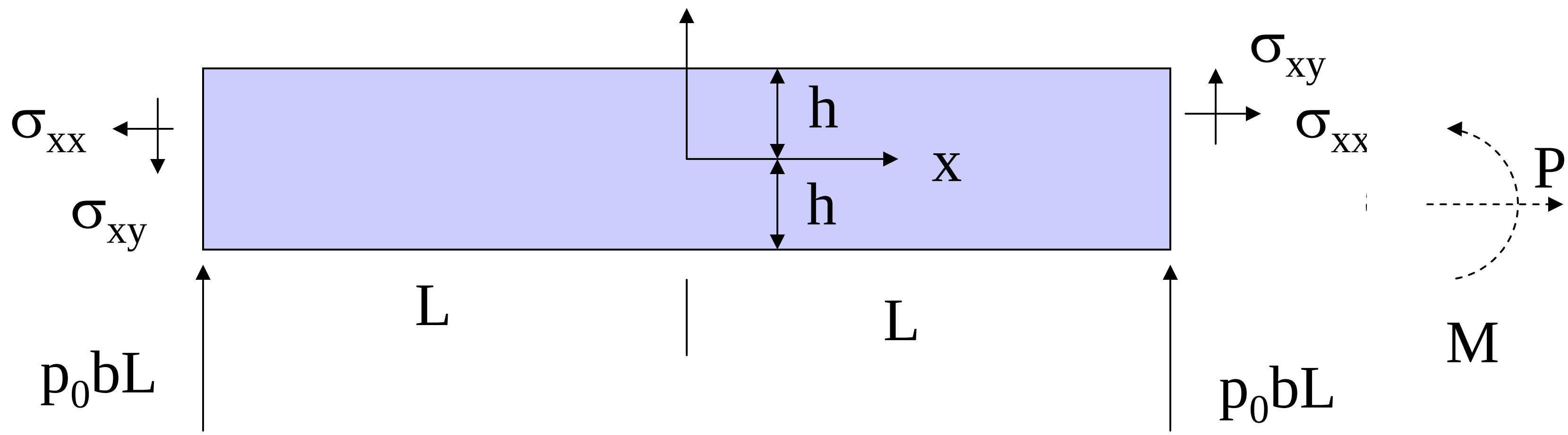
$$\sigma_{xx} = ay + \frac{3p_0}{4h^3}x^2y - \frac{p_0}{2h^3}y^3$$

$$\sigma_{xy} = \frac{3p_0}{4h}x - \frac{3p_0}{4h^3}xy^2$$

$$\sigma_{yy} = \frac{-p_0}{2} - \frac{3p_0}{4h}y + \frac{p_0}{4h^3}y^3$$

All the constants except *a* are now known.

We still have to address the boundary conditions at the ends of the beam



Saint-Venant boundary conditions

x = -L

(d) $$\int_{-h}^{+h} \sigma_{xy} b\,dy = -p_0 bL$$

(e) $$\int_{-h}^{+h} \sigma_{xx} b\,dy = 0$$

(f) $$\int_{-h}^{+h} \sigma_{xx} y b\,dy = 0$$

x = L

(a) $$\int_{-h}^{+h} \sigma_{xy} b\,dy = p_0 bL$$

(b) $$\int_{-h}^{+h} \sigma_{xx} b\,dy = 0$$ (P = 0)

(c) $$\int_{-h}^{+h} \sigma_{xx} y b\,dy = 0$$ (M = 0)

(a) : $$\int_{-h}^{+h} \sigma_{xy} b dy = p_0 bL \quad \Rightarrow \quad \frac{3p_0L}{4h}\int_{-h}^{+h}\left(1-\frac{y^2}{h^2}\right)dy = p_0 bL$$

satisfied identically

(b) : $$\int_{-h}^{+h} \sigma_{xx} b dy = 0$$

satisfied automatically
(σ_{xx} is odd in y)

(c) : $$\int_{-h}^{+h} \sigma_{xx} y b dy = 0 \quad \Rightarrow \quad a = p_0\left(\frac{3}{10h}-\frac{3L^2}{4h^3}\right)$$

(d), (e), (f) are all also satisfied with these same constants

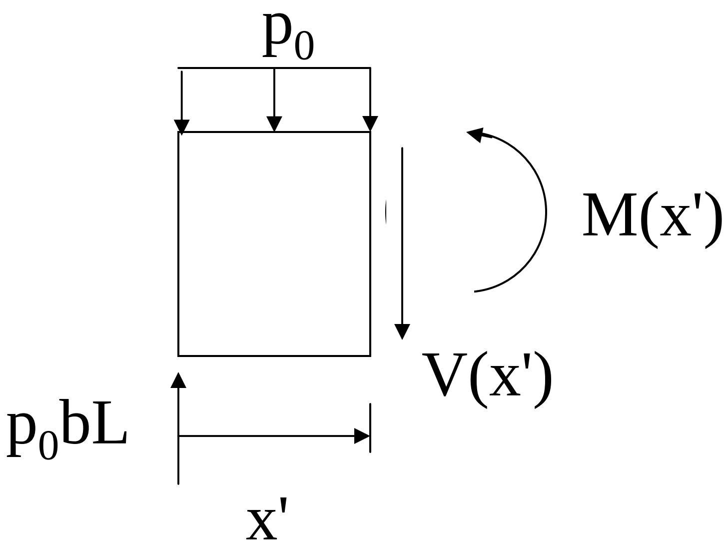
Final solution:

$$\sigma_{xx} = p_0\left(\frac{3y}{10h} - \frac{3L^2 y}{4h^3}\right) + \frac{3p_0}{4h^3}x^2 y - \frac{p_0}{2h^3}y^3$$

$$\sigma_{xy} = \frac{3p_0}{4h}x - \frac{3p_0}{4h^3}xy^2$$

$$\sigma_{yy} = \frac{-p_0}{2} - \frac{3p_0}{4h}y + \frac{p_0}{4h^3}y^3$$

Strength of materials solution in terms of x' = x + L



$$M(x') = p_0 b\left[Lx' - \frac{(x')^2}{2}\right]$$

$$= p_0 b\left(\frac{L^2}{2} - \frac{x^2}{2}\right)$$

$$V(x') = p_0 b(L - x')$$

$$= -p_0 b x$$

$$Q(y) = \frac{b(h^2 - y^2)}{2}$$

$$I = \frac{2}{3} b h^3$$

$$\sigma_{xx} = -\frac{M(x')y}{I}$$

$$\sigma_{xy} = -\frac{V(x')Q(y)}{I\,b} \quad \uparrow +$$

$$\sigma_{yy} = 0$$

Our present solution:

$$\sigma_{xx} = -\frac{M(x')\,y}{I} + p_0\left(\frac{3y}{10h} - \frac{y^3}{2h^3}\right)$$

$$\sigma_{xy} = -\frac{V(x')\,Q(y)}{Ib} \qquad \uparrow +$$

$$\sigma_{yy} = -p_0\left(\frac{1}{2} + \frac{3y}{4h} - \frac{y^3}{4h^3}\right)$$

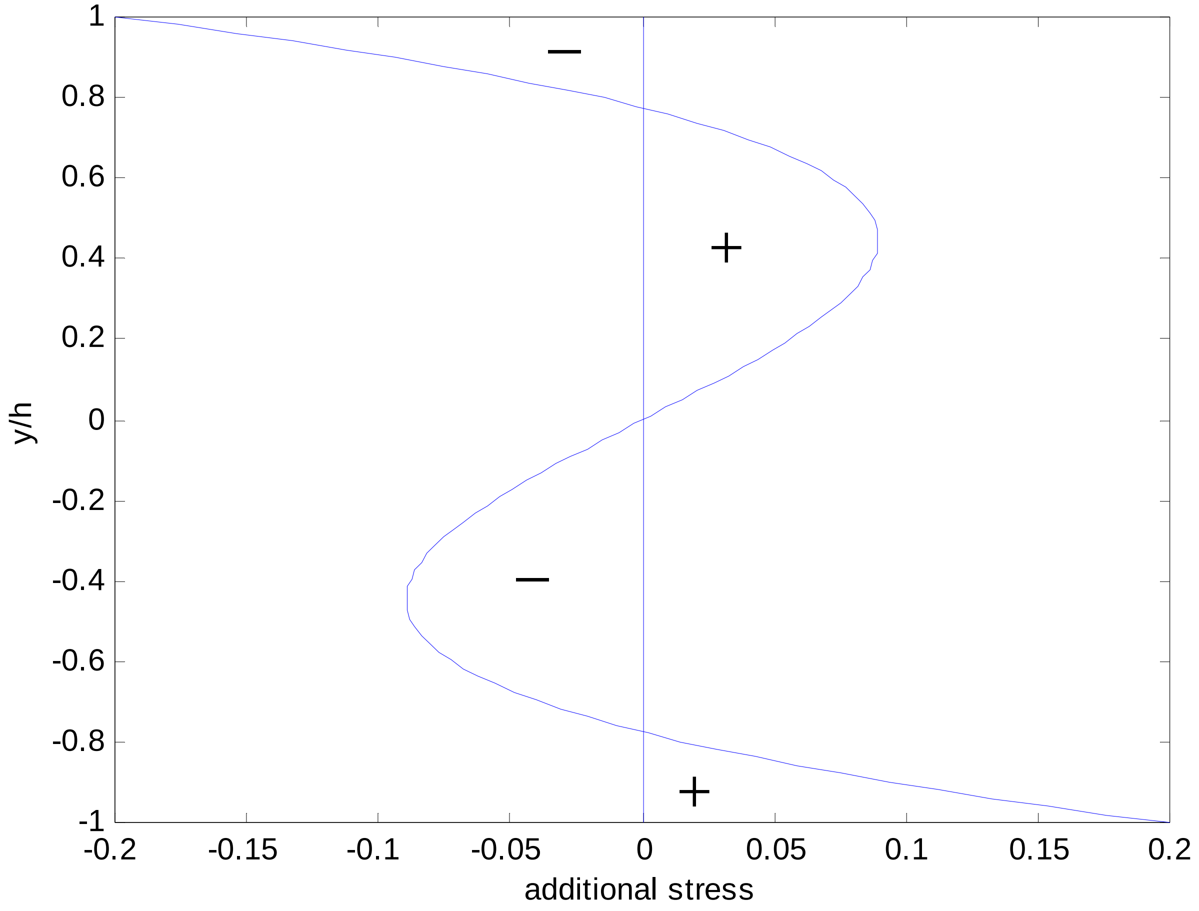
$$\sigma_{xx}\Big|_{\max}^{strength} = \frac{3p_0L^2}{4h^2}$$ At x = L (center of beam)

$$\sigma_{xx}\Big|_{\max}^{add} = \frac{p_0}{5} \qquad \sigma_{yy}\Big|_{\max}^{add} = p_0$$

$$\Rightarrow \quad \frac{\sigma_{xx}\Big|_{\max}^{add}}{\sigma_{xx}\Big|_{\max}^{strength}} = \frac{4}{15}\frac{h^2}{L^2} \qquad \frac{\sigma_{yy}\Big|_{\max}^{add}}{\sigma_{xx}\Big|_{\max}^{strength}} = \frac{4}{3}\frac{h^2}{L^2}$$

The additional flexure stress
(self equilibrated: P = M =0)

$$\frac{\sigma_{xx}}{p_0} = \left(\frac{6y}{20h} - \frac{y^3}{2h^3} \right)$$



The additional stress $$\frac{\sigma_{yy}}{p_0} = -\left(\frac{1}{2} + \frac{3y}{4h} - \frac{y^3}{4h^3}\right)$$