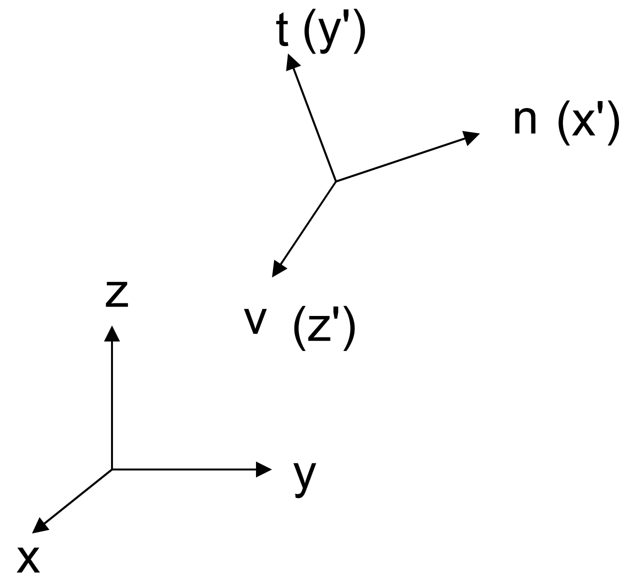
We found that stresses and strains transform according to the same rules, namely

$$[\sigma'] = [l]^T [\sigma][l]$$

$$[e'] = [l]^T [e][l]$$



where [l] is the matrix of direction cosines

$$[l] = \begin{bmatrix} n_x & t_x & v_x \\ n_y & t_y & v_y \\ n_z & t_z & v_z \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

For an isotropic elastic solid, we do not have to worry about transforming the 6x6 matrix of elastic constants when we change coordinates because those constants do not change (in fact, that is what we mean by isotropic!). However, for anisotropic materials, that is not the case. In most cases, the constants are given in a particular set of material coordinates that reflect some intrinsic symmetry directions for the material. If we use any other coordinate system, the constants in the 6x6 matrix will change.

The transformation rules for the 6x6 elastic constants are:

$$\left[C'\right] = \left[M\right]\left[C\right]\left[M\right]^T$$

where the 6x6 matrix *M* is given by

$$M = \begin{bmatrix} l_{11}^2 & l_{21}^2 & l_{31}^2 & 2l_{21}l_{11} & 2l_{31}l_{11} & 2l_{21}l_{31} \\ l_{12}^2 & l_{22}^2 & l_{32}^2 & 2l_{22}l_{12} & 2l_{12}l_{32} & 2l_{32}l_{22} \\ l_{13}^2 & l_{23}^2 & l_{33}^2 & 2l_{23}l_{13} & 2l_{13}l_{33} & 2l_{32}l_{33} \\ l_{12}l_{11} & l_{21}l_{22} & l_{31}l_{32} & l_{11}l_{22} + l_{21}l_{12} & l_{31}l_{12} + l_{11}l_{32} & l_{31}l_{22} + l_{21}l_{32} \\ l_{11}l_{13} & l_{21}l_{23} & l_{31}l_{33} & l_{13}l_{21} + l_{23}l_{11} & l_{11}l_{33} + l_{31}l_{13} & l_{33}l_{21} + l_{23}l_{31} \\ l_{13}l_{12} & l_{23}l_{22} & l_{33}l_{32} & l_{13}l_{22} + l_{23}l_{12} & l_{12}l_{33} + l_{13}l_{32} & l_{33}l_{22} + l_{23}l_{32} \end{bmatrix}$$

Example: suppose we start with a state of strain (in μ strain)

strain =

$$\begin{matrix} 300 & 50 & 20 \\ 50 & 200 & 30 \\ 20 & 30 & 100 \end{matrix}$$

Consider an orthotropic material where

C11 = 103 GPa

C22 = 50

C33 = 75

C12 = 55

C13 = 25

C23 = 40

C44 = 27.6

C55 = 10

C66 = 45

Then the elastic constants (stiffness matrix) is

$$C = \begin{matrix} 0.1030 & 0.0550 & 0.0250 & 0 & 0 & 0 \\ 0.0550 & 0.0500 & 0.0400 & 0 & 0 & 0 \\ 0.0250 & 0.0400 & 0.0750 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0276 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0100 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0450 \end{matrix}$$

in units of MPa/μ strain

If we change the strain matrix to a strain vector e ,we have

$$\text{strain} = \begin{matrix} 300 & 50 & 20 \\ 50 & 200 & 30 \\ 20 & 30 & 100 \end{matrix} \quad \Rightarrow \quad e = \begin{matrix} 300 \\ 200 \\ 100 \\ 100 \\ 40 \\ 60 \end{matrix}$$

engineering shear strains

So now we can multiply these strains by the elastic stiffness matrix to get the stresses:

$$\{\sigma\} = [C]\{e\}$$

stresses in vector form

stress_v =

44.4000
30.5000
23.0000
2.7600
0.4000
2.7000

$\Rightarrow$

stress =

$$\begin{matrix} 44.4000 & 2.7600 & 0.4000 \\ 2.7600 & 30.5000 & 2.7000 \\ 0.4000 & 2.7000 & 23.0000 \end{matrix} \quad \text{MPa}$$

Now, let's find the principal stresses and principal directions for the stresses:

```
>> [pdirs, pvals] = eig(stress)

pdirs =

    0.0217   -0.1980   -0.9800
   -0.3130    0.9296   -0.1948
    0.9495    0.3110   -0.0418
```

direction cosines from x,y,z axes to principal axes

```
pvals =

   22.1190         0         0
         0   30.8154         0
         0         0   44.9656
```

state of stress in principal stress coordinates

Note that the principal stress directions are not the principal strain directions. We can see this by calculating the principal strain directions separately

```
pdirs =

    0.0217   -0.1980   -0.9800
   -0.3130    0.9296   -0.1948
    0.9495    0.3110   -0.0418


>> [pdirs2, pvals2] = eig(strain)

pdirs2 =

   -0.0322   -0.4169   -0.9084
   -0.2525    0.8827   -0.3962
    0.9670    0.2166   -0.1337

pvals2 =

   91.4995         0         0
         0  183.7468         0
         0         0  324.7537
```

principal stress directions

$\neq$

principal strain directions

principal strains

Since we have calculated the principal stress directions, we could also find the principal stresses by first finding the strains in the principal stress coordinates:

```
>> strain_new = pdirs' *strain*pdirs

strain_new =

   92.2052   -5.9178   -6.8183
   -5.9178  190.7245  -31.8258
   -6.8183  -31.8258  317.0703
```

direction cosine matrix [l] from x,y,z axes to principal stress axes

and then we could use the stress-strain relations in those principal coordinates. But note that we need to first calculate the stiffness matrix in the principal coordinates by transforming it appropriately as we discussed previously:

$$[Cnew]=[M][C][M]^T$$

contains products of [l] components

This new stiffness matrix is:

Cnew =

0.0845	0.0281	0.0280	-0.0120	0.0032	-0.0001
0.0281	0.0677	0.0502	0.0179	-0.0062	0.0076
0.0280	0.0502	0.1033	-0.0075	0.0025	-0.0002
-0.0120	0.0179	-0.0075	0.0322	-0.0041	-0.0063
0.0032	-0.0062	0.0025	-0.0041	0.0126	-0.0038
-0.0001	0.0076	-0.0002	-0.0063	-0.0038	0.0241

which is quite different from our original stiffness matrix

C =

0.1030	0.0550	0.0250	0	0	0
0.0550	0.0500	0.0400	0	0	0
0.0250	0.0400	0.0750	0	0	0
0	0	0	0.0276	0	0
0	0	0	0	0.0100	0
0	0	0	0	0	0.0450

With this new stiffness matrix and the strains in principal coordinates we can calculate the principal stresses from our stress-strain relations

```
strain_new =

   92.2052   -5.9178   -6.8183
   -5.9178  190.7245  -31.8258
   -6.8183  -31.8258  317.0703
```

```
ev =

   92.2052
  190.7245
  317.0703
  -11.8356
  -13.6365
  -63.6516
```

engineering shears again

```
>> stress_p = Cnew*ev

stress_p =

   22.1190
   30.8154
   44.9656
    0.0000
   -0.0000
    0.0000
```

which agrees with our previous results:

```
pvals =

   22.1190         0         0
         0   30.8154         0
         0         0   44.9656
```

Now, suppose we start out with a stiffness matrix for an isotropic material:

C11 = C22 = C33 = 50 GPa
C12 = C13 = C23 = 20 GPa
C44 = C55 = C66 = (C11 - C12)/2 = 15 GPa

C2 =

0.0500	0.0200	0.0200	0	0	0
0.0200	0.0500	0.0200	0	0	0
0.0200	0.0200	0.0500	0	0	0
0	0	0	0.0150	0	0
0	0	0	0	0.0150	0
0	0	0	0	0	0.0150

MPa /μ strain

isotopic

Now, calculate this stiffness matrix in the principal stress directions found previously:

$$[Cp]=[M]^T[C2][M]$$

contains products of pdir direction cosines

We find: Cp =

0.0500	0.0200	0.0200	0.0000	-0.0000	0.0000
0.0200	0.0500	0.0200	0.0000	0	0
0.0200	0.0200	0.0500	0.0000	-0.0000	0.0000
0.0000	0.0000	0.0000	0.0150	0	-0.0000
-0.0000	-0.0000	-0.0000	0	0.0150	0.0000
0.0000	0.0000	0.0000	-0.0000	0.0000	0.0150

so the matrix has not changed, and will not change as measured in any other set of orthogonal coordinate directions either.