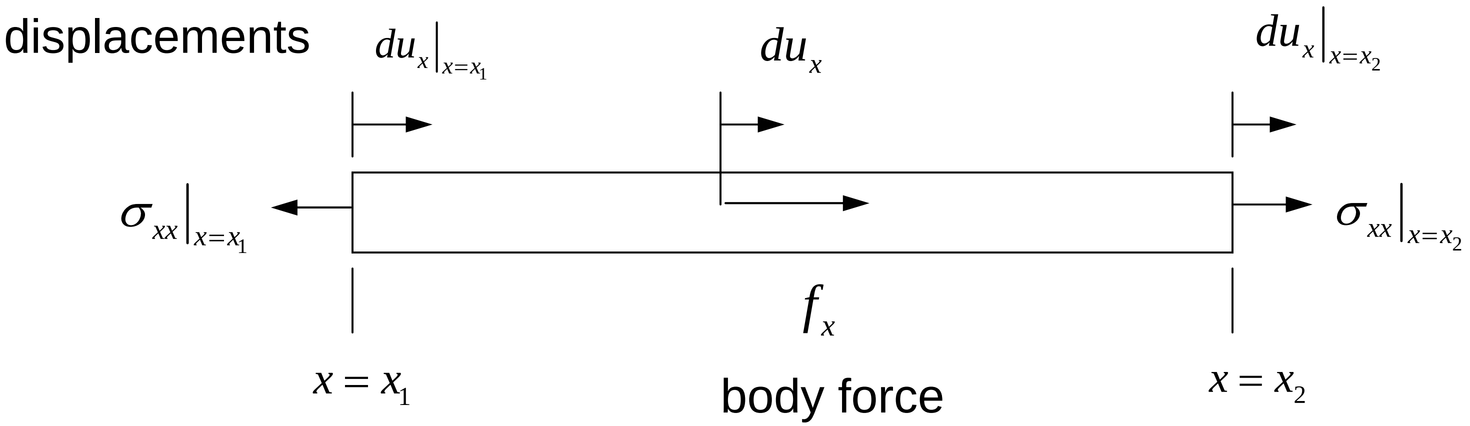
Work- Energy

1-D example



A = cross-sectional area

Work done by the stresses and body force

$$dW = \int_A \sigma_{xx} du_x \Big|_{x=x_1}^{x=x_2} dA + \int_{x_1}^{x_2}\int_A f_x du_x \, dA\, dx$$

$$= \int_{x_1}^{x_2}\int_A \frac{d}{dx}\left(\sigma_{xx} du_x\right) dA\, dx + \int_{x_1}^{x_2}\int_A f_x du_x \, dA\, dx$$

$$= \int_{x_1}^{x_2}\int_A \underbrace{\left(\frac{d\sigma_{xx}}{dx} + f_x\right)}_{= 0 \text{ by equilibrium}} du_x dA\, dx + \int_{x_1}^{x_2}\int_A \sigma_{xx} \underbrace{d\left(\frac{du_x}{dx}\right)}_{= \, de_{xx}} dA\, dx$$

$$dW = \int_{x_1}^{x_2}\int_A \sigma_{xx} de_{xx} dA dx = \int_V \sigma_{xx} de_{xx} dV$$

V … volume

work done = increase in internal energy (strain energy)

$$dW = dU$$

so

$$dU = \int_V \sigma_{xx} de_{xx} dV = \int_V du_0 dV$$

du_0 … increase in strain energy density (strain energy/vol)

$$du_0 = \sigma_{xx} de_{xx}$$

Now, if $\sigma_{xx} = f\left(e_{xx}\right)$ $\quad du_0 = du_0\left(e_{xx}\right) = \dfrac{du_0}{de_{xx}} de_{xx}$

which gives

$$\sigma_{xx} = \frac{du_0}{de_{xx}} \longleftarrow$$

true for both linear and non-linear elastic materials

For a <u>linear</u> elastic material

$$\sigma_{xx} = E\, e_{xx}$$

so $$du_0 = E e_{xx} de_{xx}$$

$$u_0 = \frac{E e_{xx}^2}{2} + \cancelto{}{const}$$

Different forms of u_0

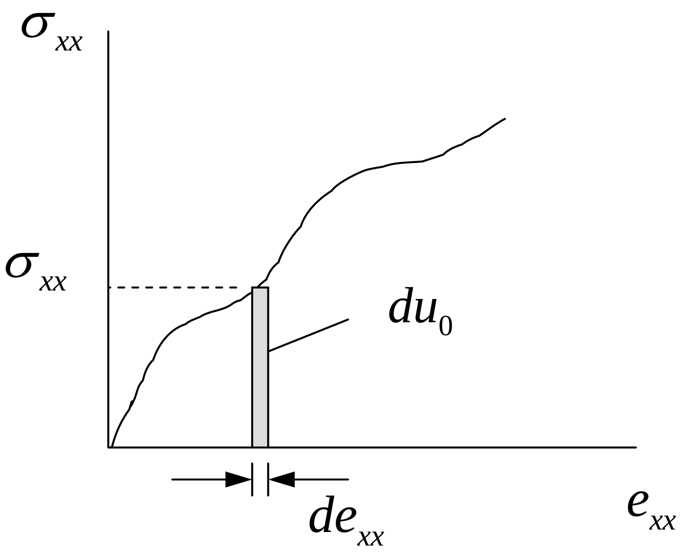
$$u_0\left(e_{xx}\right) = \frac{E e_{xx}^2}{2}$$

$$u_0\left(\sigma_{xx}\right) = \frac{\sigma_{xx}^2}{2E}$$

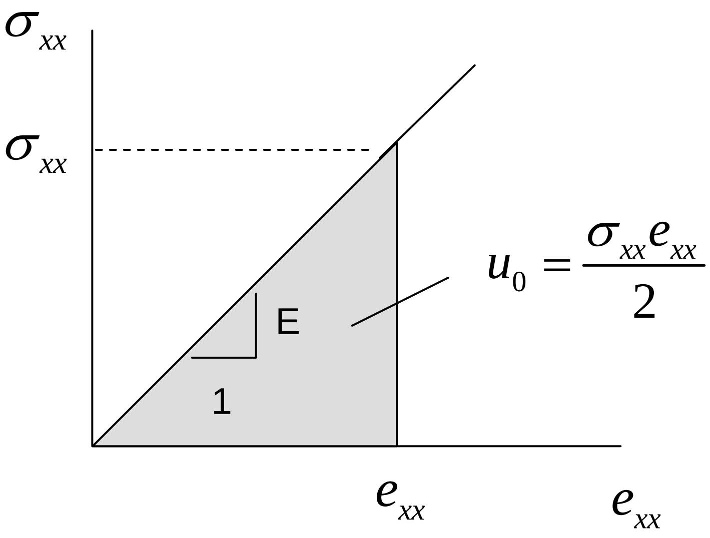
$$u_0\left(\sigma_{xx}, e_{xx}\right) = \frac{\sigma_{xx} e_{xx}}{2}$$

Strain energy density and the stress-strain curve

In general



For linear elastic material



Complimentary strain energy density, u_0^c

Definition: $u_0^c = \sigma_{xx} e_{xx} - u_0$

Thus, $du_0^c = \cancel{\sigma_{xx} de_{xx}} + e_{xx} d\sigma_{xx} - \cancel{du_0}$

$$du_0^c = e_{xx} d\sigma_{xx}$$

Now, if $e_{xx} = g\left(\sigma_{xx}\right)$ $\quad u_0^c = u_0^c\left(\sigma_{xx}\right) = \frac{du_0^c}{d\sigma_{xx}} d\sigma_{xx}$

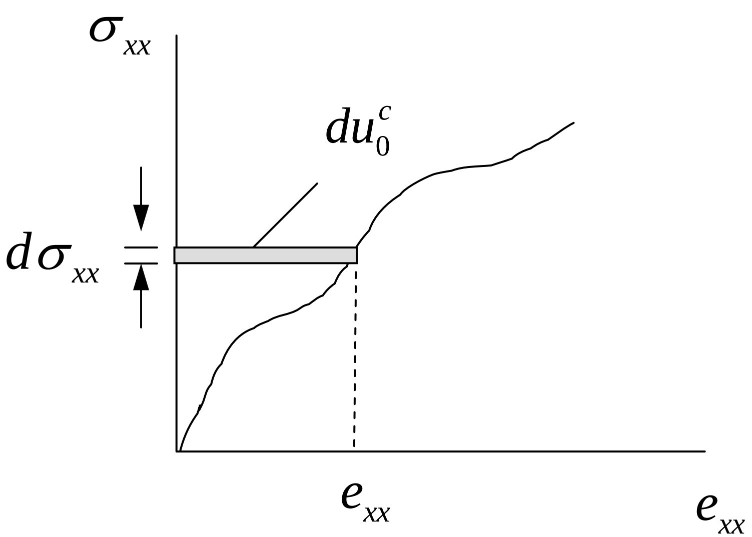
which gives $e_{xx} = \frac{du_0^c}{d\sigma_{xx}}$

For a linear elastic material $u_0^c = \sigma_{xx} e_{xx} - \frac{\sigma_{xx} e_{xx}}{2}$

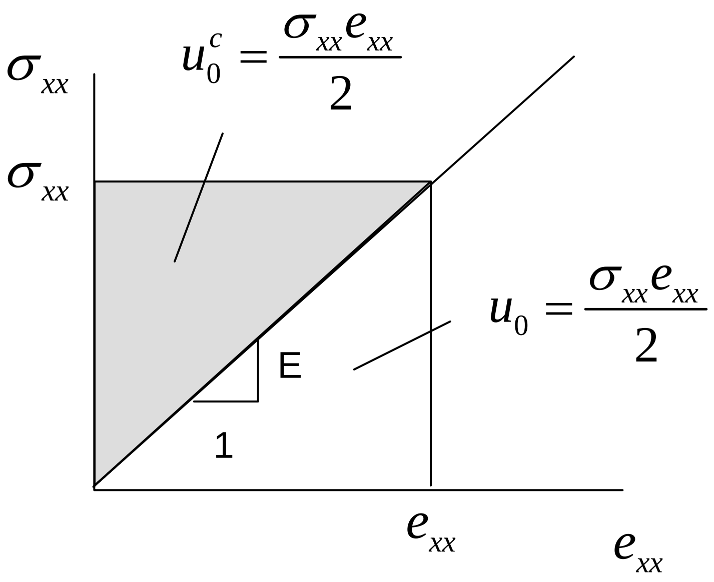
$$= \frac{\sigma_{xx} e_{xx}}{2} = u_0$$

Complimentary strain energy density and the stress-strain curve

In general



For linear elastic material



Generalizations to 3-D stresses and strains

1-D

$$dW = \int_A \sigma_{xx} du_x \Big|_{x=x_1}^{x=x_2} dA + \int_{x_1}^{x_2} \int_A f_x du_x \, dA \, dx$$

3-D

$$dW = \int_S \mathbf{T}^{(n)} \cdot d\mathbf{u} \, dS + \int_V \mathbf{f} \cdot d\mathbf{u} \, dV$$

work done by surface tractions on surface, S

work done by body forces over volume, V

for both 1-D and 3-D

$$dW = dU = \int_V du_0 dV$$

1-D

$$du_0 = \sigma_{xx} de_{xx}$$

3-D

$$du_0 = \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma_{ij} de_{ij}$$

1-D for $u_0\left(e_{xx}\right)$

3-D for $u_0\left(e_{ij}\right)$

$$\sigma_{xx} = \frac{du_0}{de_{xx}}$$

$$\sigma_{ij} = \frac{\partial u_0}{\partial e_{ij}} \qquad \begin{pmatrix} i = 1,2,3 \\ j = 1,2,3 \end{pmatrix}$$

For a linear elastic material

1-D $\sigma_{xx} = Ee_{xx}$ or $e_{xx} = \dfrac{\sigma_{xx}}{E}$

$$u_0\left(e_{xx}\right) = \frac{Ee_{xx}^2}{2}$$

$$\Rightarrow \quad u_0\left(\sigma_{xx}\right) = \frac{\sigma_{xx}^2}{2E}$$

$$u_0\left(\sigma_{xx}, e_{xx}\right) = \frac{\sigma_{xx}e_{xx}}{2}$$

3-D

$$\sigma_{ij} = \sum_{k=1}^{3}\sum_{l=1}^{3} C_{ijkl} e_{kl} \qquad \begin{pmatrix} i = 1,2,3 \\ j = 1,2,3 \end{pmatrix}$$

or

$$e_{ij} = \sum_{k=1}^{3}\sum_{l=1}^{3} D_{ijkl} \sigma_{kl} \qquad \begin{pmatrix} i = 1,2,3 \\ j = 1,2,3 \end{pmatrix}$$

$\Longrightarrow$

$$u_0\left(e_{ij}\right) = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} C_{ijkl} e_{ij} e_{kl}$$

$$u_0\left(\sigma_{ij}\right) = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} D_{ijkl} \sigma_{ij} \sigma_{kl}$$

$$u_0\left(\sigma_{ij}, e_{ij}\right) = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \sigma_{ij} e_{ij}$$

Complimentary strain energy density

1-D

$$u_0^c = \sigma_{xx} e_{xx} - u_0$$

$$du_0^c = e_{xx} d\sigma_{xx}$$

$$e_{xx} = \frac{du_0^c}{d\sigma_{xx}}$$

3-D

$$u_0^c = \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma_{ij} e_{ij} - u_0$$

$$du_0^c = \sum_{i=1}^{3} \sum_{j=1}^{3} e_{ij} d\sigma_{ij}$$

$$e_{ij} = \frac{\partial u_0^c}{\partial \sigma_{ij}} \qquad \begin{pmatrix} i = 1, 2, 3 \\ j = 1, 2, 3 \end{pmatrix}$$

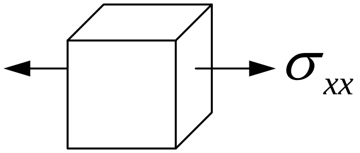
For linear elastic material

1-D or 3-D

$$u_0^c = u_0$$

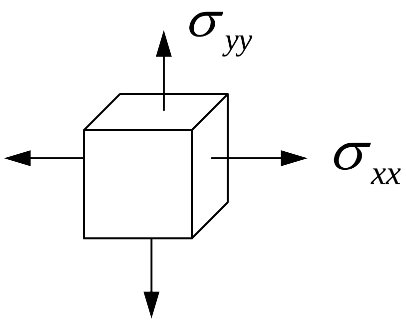
Strain energy density for a linear elastic, isotropic material

(1) apply σ_{xx}



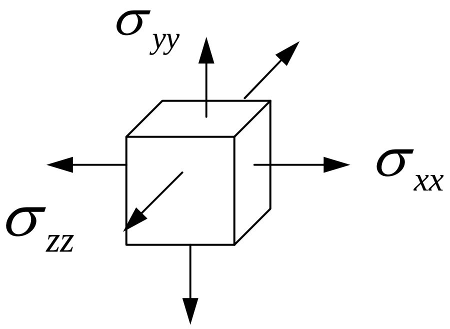
$$u_0^{(1)} = \frac{\sigma_{xx} e_{xx}}{2} = \frac{\sigma_{xx}^2}{2E}$$

(2) apply σ_{yy}, holding σ_{xx} fixed



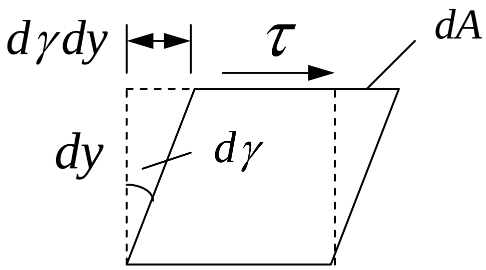
$$u_0^{(2)} = \frac{\sigma_{yy}^2}{2E} + \sigma_{xx} \underbrace{\left(\frac{-\nu \sigma_{yy}}{E} \right)}_{\text{Poisson strain in x-direction}}$$

(3) apply σ_{zz}, holding σ_{xx}, σ_{yy} fixed



$$u_0^{(3)} = \frac{\sigma_{zz}^2}{2E} + \sigma_{xx} \left(\frac{-\nu \sigma_{zz}}{E} \right) + \sigma_{yy} \left(\frac{-\nu \sigma_{zz}}{E} \right)$$

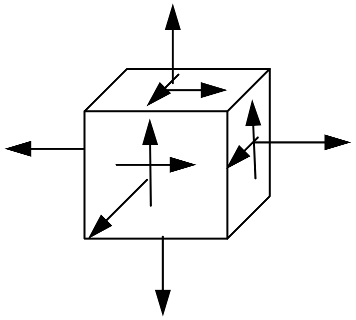
For a shear strain acting on a face of area A



$$dW = \int \tau dA\left(d\gamma dy\right) = \int_V \tau d\gamma \, dV = \int_V du_0 dV \quad \Rightarrow \quad du_0 = \tau d\gamma$$

With $\tau = G\gamma$ we obtain $u_0 = \dfrac{\tau^2}{2G} = \dfrac{G\gamma^2}{2} = \dfrac{\tau\,\gamma}{2}$

(4) now apply all three shear stresses $\sigma_{xy}, \sigma_{xz}, \sigma_{yz}$



$$u_0^{(4)} = \frac{\sigma_{xy}^2}{2G} + \frac{\sigma_{xz}^2}{2G} + \frac{\sigma_{yz}^2}{2G}$$

Adding up all these contributions to the strain energy density

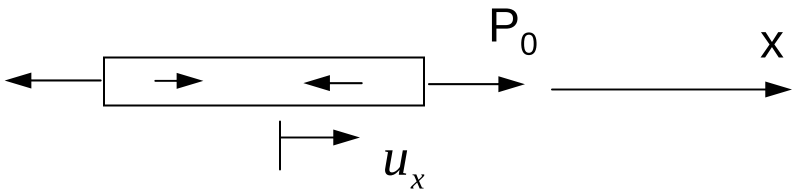
$$
\begin{aligned}
u_0\left(\sigma_{ij}\right) &= u_0^{(1)} + u_0^{(2)} + u_0^{(3)} + u_0^{(4)} \\
&= \frac{1}{2E}\left[\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 - 2\nu\left(\sigma_{xx}\sigma_{yy} + \sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz}\right)\right] \\
&\quad + \frac{1}{2G}\left[\sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2\right]
\end{aligned}
$$

In terms of the strains

$$
\begin{aligned}
u_0\left(e_{ij}\right) &= \frac{E}{2\left(1+\nu\right)\left(1-2\nu\right)}\left[\left(1-\nu\right)\left(e_{xx}^2 + e_{yy}^2 + e_{zz}^2\right) + 2\nu\left(e_{xx}e_{yy} + e_{xx}e_{zz} + e_{yy}e_{zz}\right)\right] \\
&\quad + \frac{G}{2}\left[\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2\right]
\end{aligned}
$$

Total Strain Energy Expressions for Various Loadings

Axial Loads

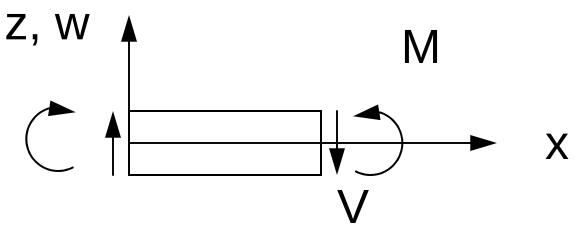


$$\sigma_{xx} = \frac{P(x)}{A}, \quad e_{xx} = \frac{du_x}{dx}, \quad \sigma_{xx} = Ee_{xx}$$

$$U(P) = \frac{1}{2}\int_0^L \frac{\sigma_{xx}^2}{E} A dx = \frac{1}{2}\int_0^L \frac{[P(x)]^2}{AE} dx$$

$$U(u_x) = \frac{1}{2}\int_0^L Ee_{xx}^2 A dx = \frac{1}{2}\int_0^L EA\left[\frac{du_x}{dx}\right]^2 dx$$

Bending



1. flexure stresses

$$\sigma_{xx} = \frac{-M(x)\,z}{I}, \quad e_{xx} = \frac{-M(x)\,z}{EI} = -\frac{d^2 w(x)}{dx^2}\,z$$

$$U(M) = \int_0^L \int_A \frac{\sigma_{xx}^2}{2E}\,dA\,dx = \frac{1}{2}\int_0^L \left\{\int_A z^2\,dA\right\} \frac{\left[M(x)\right]^2}{EI^2}\,dx = \frac{1}{2}\int_0^L \frac{\left[M(x)\right]^2}{EI}\,dx$$

$$U(w) = \int_0^L \int_A \frac{E e_{xx}^2}{2}\,dA\,dx = \frac{1}{2}\int_0^L \left\{\int_A z^2\,dA\right\} E\left[\frac{d^2 w(x)}{dx^2}\right]^2 dx = \frac{1}{2}\int_0^L EI\left[\frac{d^2 w(x)}{dx^2}\right]^2 dx$$

2. Bending shear stress

$$\tau = \frac{V(x)Q(z)}{I\,t(z)}$$

$$U(V) = \int_0^L \int_A \frac{\tau^2}{2G}\,dA\,dx = \frac{1}{2}\int_0^L \left\{ \frac{A}{I^2} \int_A \left[\frac{Q(z)}{t(z)} \right]^2 dA \right\} \frac{\left[V(x)\right]^2}{GA}\,dx$$

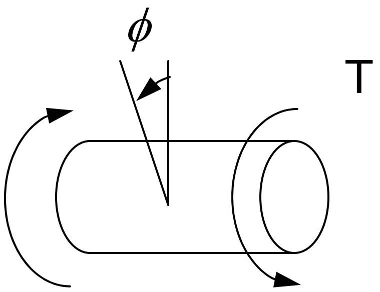
$$= \frac{1}{2}\int_0^L k \frac{\left[V(x)\right]^2}{GA}\,dx$$

$$k = \frac{A}{I^2}\int_A \left[\frac{Q(z)}{t(z)} \right]^2 dA$$

k = 1.2 for rectangular cross-section

This strain energy is much smaller than that due to flexure for long, slender beams and is often neglected

Torsion



$$\tau = \frac{T(x)\, r}{J}$$

$$\gamma = \frac{d\phi(x)}{dx} r$$

$$\tau = G\,\gamma$$

$$U(T) = \int_0^L \int_A \frac{\tau^2}{2G} dA dx = \frac{1}{2} \int_0^L \left\{ \int_A r^2 dA \right\} \frac{\left[T(x)\right]^2}{GJ^2} dx = \frac{1}{2} \int_0^L \frac{\left[T(x)\right]^2}{GJ} dx$$

$$U(\phi) = \int_0^L \int_A \frac{G\gamma^2}{2} dA dx = \frac{1}{2} \int_0^L \left\{ \int_A r^2 dA \right\} G \left[\frac{d\phi(x)}{dx} \right]^2 dx = \frac{1}{2} \int_0^L GJ \left[\frac{d\phi(x)}{dx} \right]^2 dx$$