Strain Energy Methods

Principles of Virtual Work and Stationary Potential Energy

Consider an elastic body which carries a set of loads, some if which are distributed and some of which are discrete (concentrated forces and moments). Let the deflection at the concentrated force P_i $(i = 1,...,N)$ in the direction of that force be given by Δ_i. Similarly, let the rotation at the concentrated moment M_i $(i = 1,...,M)$ in the direction determined from that moment by the right hand rule be given by θ_i. We will assume the strain energy of this body is expressed in terms of these displacements and rotations as $U = U(\Delta_1,...,\Delta_N,\theta_1,...,\theta_M)$.

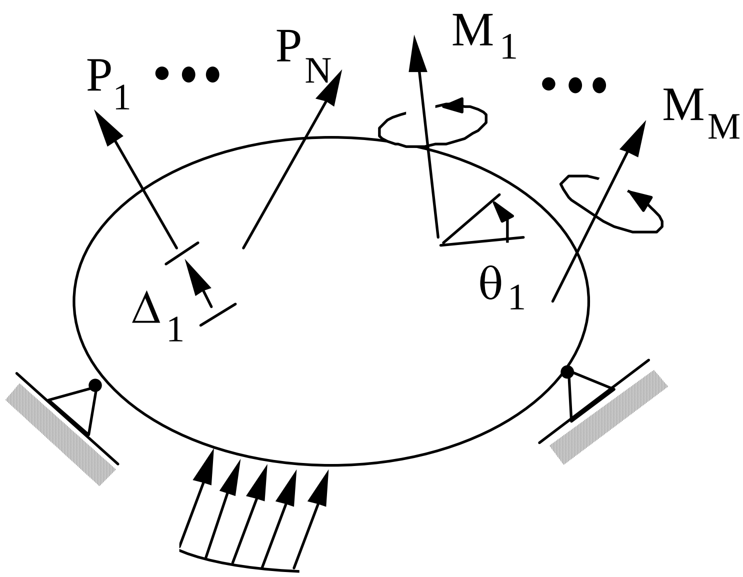


Fig.1

Now, imagine we fix the concentrated forces and moments and make small changes in these displacements and rotations $\delta\Delta_i, \delta\theta_i$. These changes in turn will cause the strain energy to change by a small amount, which we call δU. Then the principle of virtual work says that the work done by the forces and moments taken through these “virtual” displacements and rotations must be equal to the resulting“ virtual” change in the strain energy, i.e

$$\delta U = \sum_{i=1}^{N} P_i\delta\Delta_i + \sum_{i=1}^{M} M_i\delta\theta_i \tag{1}$$

This principle is often also stated in a different form , called the principle of stationary potential energy, where the potential energy, Π, is defined as

$$\Pi = U - \sum_{i=1}^{N} P_i \Delta_i - \sum_{i=1}^{M} M_i \theta_i \tag{2}$$

From the principle of virtual work it follows that for this potential energy

$$\delta\Pi = \delta U - \sum_{i=1}^{N} P_i \delta\Delta_i - \sum_{i=1}^{M} M_i \delta\theta_i = 0 \tag{3}$$

i.e. the potential energy must be *stationary*. Actually, we can state a much stronger requirement, namely that the strain energy must be also be a minimum when the body of Fig. 1 is in equilibrium, but we will not show that result here.

Castigliano's First Theorem

Since the strain energy has been assumed to be written as a function of the displacements and rotations, it follows that

$$\delta U = \sum_{i=1}^{N} \frac{\partial U}{\partial \Delta_i} \delta\Delta_i + \sum_{i=1}^{M} \frac{\partial U}{\partial \theta_i} \delta\theta_i \tag{4}$$

so that placing this relationship into either Eq. (1) or Eq.(4) gives

$$\sum_{i=1}^{N} \left(\frac{\partial U}{\partial \Delta_i} - P_i \right) \delta\Delta_i + \sum_{i=1}^{M} \left(\frac{\partial U}{\partial \theta_i} - M_i \right) \delta\theta_i = 0 \tag{5}$$

which can only be satisfied, for all possible virtual displacements and rotations, if

$$\begin{aligned} \frac{\partial U}{\partial \Delta_i} &= P_i \quad (i = 1, ..., N) \\ \frac{\partial U}{\partial \theta_i} &= M_i \quad (i = 1, ..., M) \end{aligned} \tag{6}$$

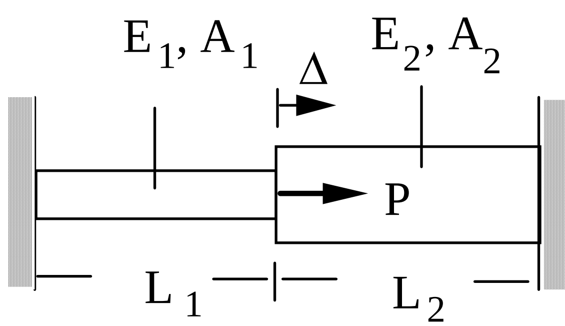
which are called *Castigliano's first theorem*.

Note the similarity of Castigliano's theorem and the result we derived previously for the strain energy density, namely

$$\frac{\partial u_0}{\partial e_{ij}} = \sigma_{ij}$$

Example of the use of Castigliano's theorem:

Consider an axial load problem where two bars are welded together between two rigid walls and are subjected to a force P at their connection



If we let Δ be the displacement at the load P and assume that the strains are constant in each bar, then the strain energy of the entire system is

$$U(\Delta) = \frac{1}{2}\frac{A_1E_1}{L_1}\Delta^2 + \frac{1}{2}\frac{A_2E_2}{L_2}\Delta^2 \tag{7}$$

Castigliano's theorem gives

$$P = \frac{\partial U}{\partial \Delta} = \left(\frac{A_1E_1}{L_1} + \frac{A_2E_2}{L_2}\right)\Delta$$

so solving for Δ we find

$$\Delta = \frac{P}{\left(\frac{A_1E_1}{L_1} + \frac{A_2E_2}{L_2}\right)} \tag{8}$$

which can be verified to be the exact result by solving this problem directly.

The Rayleigh-Ritz Method

In the previous example we obtained the exact solution because the assumed deformation field that we used to calculate the work and energy coincided with that of the exact solution. In more complicated problems it may not be simple matter to obtain exact deformation expressions. Nevertheless, if we can make a "reasonable" guess at the form of the deformations in terms of some known simple functions and unknown coefficients, then the principle of stationary potential energy (or virtual work) gives us a method to determine the unknown coefficients in such a manner that equilibrium will be approximately satisfied. In many cases, this approximate solution will closely follow the deformation of the actual solution. This method is called the Rayleigh-Ritz method. The procedure of the method is to assume that the displacements of the body can be represented parametrically in terms of a set of basis functions and n unknown parameters $(a_1, \ldots, a_n)$. For example, in a 1-D problem we might take the displacement in the form

$$u_x = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$$

In the Rayleigh-Ritz method, it is assumed that the functions and constants are chosen such that any boundary conditions involving the displacements or rotations of the body (so-called "essential" boundary conditions) are satisfied. Then the functions and coefficients are placed into the expression for the potential energy of the body so that this potential energy can be expressed in the form

$$\Pi = \Pi(a_0, \ldots, a_{n-1})$$

The potential energy is made stationary by requiring that

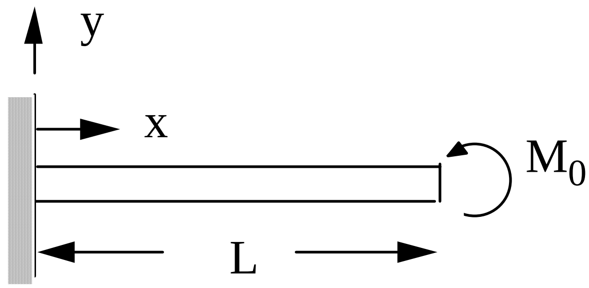
$$\delta\Pi = \sum_{i=0}^{n-1} \frac{\partial \Pi}{\partial a_i} \delta a_i = 0$$

which can only be satisfied for arbitrary changes of these coefficients if

$$\frac{\partial \Pi}{\partial a_i} = 0 \quad (i = 0, \ldots, n-1)$$

leading to n equations to be solved for the n coefficients.

As an example of the Rayleigh-Ritz method, consider a cantilever beam loaded by an end moment as shown.



The exact solution for the deflection of the beam is

$$y(x) = \frac{M_0 x^2}{2EI}$$

so that the end deflection is given by

$$y(L) = \frac{M_0 L^2}{2EI}$$

To obtain an approximate solution to this problem, let us assume

$$y(x) = a\left[1 - \cos\left(\frac{\pi x}{2L}\right)\right]$$

This function meets the requirements stated previously, namely it satisfies the essential boundary conditions in this problem which are:

$$y\Big|_{x=0} = 0$$
$$\frac{dy}{dx}\Big|_{x=0} = 0$$

The total potential energy is then given by

$$\begin{aligned}\Pi &= U - M_0 \theta\Big|_{x=l} \\ &= \frac{EI}{2}\int_0^L \left[\frac{d^2 y}{dx^2}\right]^2 dx - M_0 \frac{dy}{dx}\Big|_{x=L}\end{aligned}$$

placing the approximate deflection expression into this expression and carrying out the indicated differentiations and integrations, we find

$$\Pi = \frac{\pi^4 EIa^2}{64L^3} - \frac{\pi M_0 a}{2L}$$

Requiring that $\frac{\partial \Pi}{\partial a} = 0$ then yields $a = \frac{16 M_0 L^2}{\pi^3 EI}$

But $a = y(L)$ and

$$y(L) = \frac{0.52 M_0 L^2}{EI}$$

which is very close to the exact solution.

Although the Rayleigh-Ritz method is a very powerful tool, for complicated 3-D problems it is impossible to make good guesses for what the deformations might be for the entire body, i.e. to choose *global* functions of approximation. However, suppose the body is broken up into small elements over which *locally* the deformations can be reasonably assumed to have simple variations, and these variations are written in terms of unknown parameters (called nodal variables). Then the principle of stationary potential energy (or virtual work) can be used to form up a set of linear equations for these nodal variables (in the example just shown there was just one nodal variable $a = y(L)$). Solving this linear system yields an approximate solution for the deformation in the entire body. This is basic idea behind the *Finite Element Method*, which we will discuss briefly later.

Principles of Complimentary Virtual Work and Stationary Complimentary Potential Energy

Consider the body in Fig 1 again where concentrated loads and moments were shown together with their corresponding displacements and rotations. Now imagine these displacements and rotations are held fixed while we change the applied concentrated forces and moments by small "virtual" amounts $\delta P_i, \delta M_i$ where these virtual changes must not violate the equations of equilibrium. For a statically determinant elastic system we can write the complimentary strain energy in terms of the applied loads, i.e. $U^c = U^c(P_1,...,P_N, M_1,...,M_M)$ so that the virtual changes in the applied loads will cause this complimentary strain energy to change by an amount δU^c. The principle of complimentary virtual work states that

$$\delta U^c = \sum_{i=1}^{N} \Delta_i \delta P_i + \sum_{i=1}^{M} \theta_i \delta M_i \qquad (9)$$

If we define a complimentary potential energy, Π^c, for the system as

$$\Pi^c = U^c - \sum_{i=1}^{N} \Delta_i P_i - \sum_{i=1}^{M} \theta_i M_i$$

then the principle of complimentary virtual work can also be stated as the requirement that the complimentary potential energy must be stationary, i.e.

$$\delta\Pi^c = \delta U^c - \sum_{i=1}^{N} \Delta_i P_i - \sum_{i=1}^{M} \theta_i M_i = 0$$

Again, as with the case of stationary potential energy, we could actually prove a much stronger result, namely that this complimentary potential energy must be a minimum, but we will not do so here.

Engesser's First Theorem and Castigliano's Second Theorem

Since, as stated earlier, the body of Fig. 1 is assumed to be a statically determinant problem where the complimentary strain energy can be written explicitly in terms of the applied loads and moments only, it follows that the change in complimentary strain energy also can be expressed as

$$\delta U^c = \sum_{i=1}^{N} \frac{\partial U^c}{\partial P_i} \delta P_i + \sum_{i=1}^{M} \frac{\partial U^c}{\partial M_i} \delta M_i$$

so Eq. (9) becomes

$$\sum_{i=1}^{N} \left(\frac{\partial U^c}{\partial P_i} - \Delta_i \right) \delta P_i + \sum_{i=1}^{M} \left\{ \frac{\partial U^c}{\partial M_i} - \theta_i \right\} \delta M_i = 0 \qquad (10)$$

If the virtual changes in the applied loads are varied independently, the only way for Eq. (10) to be satisfied is if

$$\begin{aligned} \frac{\partial U^c}{\partial P_i} &= \Delta_i \quad (i = 1, ..., N) \\ \frac{\partial U^c}{\partial M_i} &= \theta_i \quad (i = 1, ..., M) \end{aligned} \qquad (11)$$

Eq. (11) is called *Engesser's first theorem*. For a *linear* elastic material, $U^c = U$ so that Engesser's theorem becomes

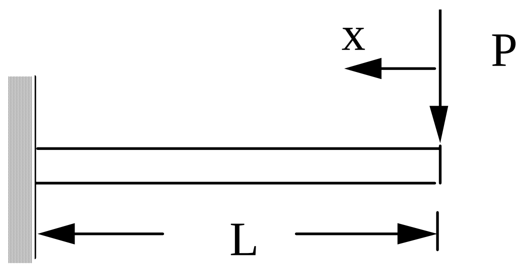
$$\begin{aligned} \frac{\partial U}{\partial P_i} &= \Delta_i \quad (i = 1, ..., N) \\ \frac{\partial U}{\partial M_i} &= \theta_i \quad (i = 1, ..., M) \end{aligned} \qquad (12)$$

which is usually referred to as *Castigliano's second theorem*.

Again, note the similarity between Engesser's first theorem and the result we had earlier for the complimentary strain energy density in terms of strains and stresses, namely

$$e_{ij} = \frac{\partial u_0^c}{\partial \sigma_{ij}}$$

Example1. The use of Castigliano's second theorem



Consider the following cantilever beam problem

The bending moment in the beam is given by $M(x) = -Px$ so that the strain energy is

$$U = \frac{1}{2}\int_0^L \frac{[M(x)]^2}{EI}\,dx$$
$$= \frac{1}{6}\frac{P^2 L^3}{EI}$$

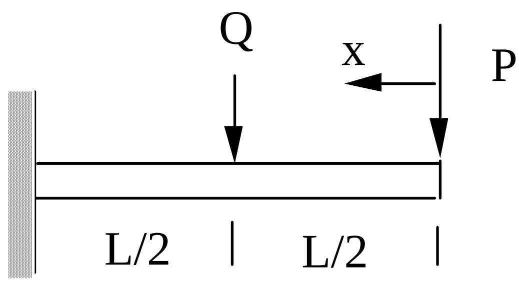
Castigliano's second theorem says the deflection at the load P in the direction of P is given by

$$\Delta_P = \frac{\partial U}{\partial P} = \frac{1}{3}\frac{PL^3}{EI}$$

which can be verified independently by equilibrium methods.

Example 2. Use of the dummy load method

As the above example showed, we can obtain a deflection (or rotation) at the location of any applied force (or moment). However, we can use the concept of a dummy load to obtain the deflection or rotation at any point in the body. To see this, consider our cantilever beam problem again where now we want to obtain the deflection at the center of the beam. In order to use Castigliano's second theorem, we place a dummy load, Q, at the center, as shown



The bending moment in the beam is now

$$\begin{aligned} M(x) &= -Px \quad (0 < x < L/2) \\ &= -Px - Q(x - L/2) \quad (L/2 < x < L) \end{aligned}$$

so that the strain energy is

$$\begin{aligned} U &= U(P,Q) \\ &= \frac{1}{2EI}\left\{ \int_0^{L/2} P^2 x^2 dx + \int_{L/2}^{L} \left[Px + Q(x - L/2)\right]^2 dx \right\} \end{aligned}$$

from Castgliano's second theorem, then we have

$$\Delta_Q = \frac{\partial U(P,Q)}{\partial Q} = \frac{1}{EI}\int_{L/2}^{L} \left[Px + Q(x - L/2)\right](x - L/2)dx \tag{13}$$

This is the deflection at Q due to both P and Q, whereas we want the deflection at Q due to P only, which we can obtain from Eq. (13) by simply setting Q = 0 to find

$$\begin{aligned} \Delta_Q\Big|_{Q=0} &= \left.\frac{\partial U(P,Q)}{\partial Q}\right|_{Q=0} = \frac{1}{EI}\int_{L/2}^{L} (Px)(x - L/2)dx \\ &= \frac{5}{48}\frac{PL^3}{EI} \end{aligned} \tag{14}$$

By using a dummy moment instead of a dummy force, the same procedure would have allowed us to obtain the local rotation at any point in the beam.

There is an alternate way of viewing this dummy load process which leads to what is called the *unit load method*. Consider, for example, a bending problem of the type we just considered where the beam is acted upon by a set of applied forces P_i and moments M_i. Following the dummy load procedure we calculate

$$\Delta_Q\big|_{Q=0} = \left.\frac{\partial U(Q, P_i, M_i)}{\partial Q}\right|_{Q=0}$$

where U is given by

$$U = \frac{1}{2EI}\int_0^L M^2 dx$$

so that

$$\Delta_Q\big|_{Q=0} = \frac{1}{EI}\int_0^L M\big|_{Q=0} \left.\frac{\partial M}{\partial Q}\right|_{Q=0} dx \qquad (15)$$

But $M\big|_{Q=0} = M(P_i, M_i, x)$ is just the moment distribution without the dummy load present and by superposition, we have the total moment distribution is

$$M(Q, P_i, M_i, x) = M(P_i, M_i, x) + M_1(Q, x) \qquad (16)$$

where $M_1(Q,x)$ is the moment distribution due to Q only. Since we are dealing with a linear system

$$M_1(Q,x) = QM_1(Q=1,x) \qquad (17)$$

where $M_1(Q=1,x)$ is the moment distribution due to a *unit* force at Q only. Using Eqs. (16) and (17), it follows that

$$\left.\frac{\partial M(Q, P_i, M_i, x)}{\partial Q}\right|_{Q=0} = M_1(Q=1, x)$$

so that Eq.(14) becomes

$$\Delta_Q\big|_{Q=0} = \frac{1}{EI}\int_0^L M(P_i, M_1, x) M_1(Q=1, x) dx \qquad (18)$$

which is an expression for the desired displacement using the unit load method.

The advantage of using Eq.(18) over the original dummy load procedure is that in the unit load method we only need to calculate our original bending moment distribution (without Q) and the bending moment from a unit load by itself rather than having to compute a bending moment when the original loads and Q are simultaneously present. As you can verify yourself, having both the original loads and Q present leads to more complicated moment distributions expressions where much of the complexity ultimately disappears anyway (through the differentiation process and subsequently setting Q = 0) in the original dummy load method. Note that we can use a unit moment in exactly the same way as done here with a unit force to calculate the rotation at any point in the beam instead.

Example 3. The use of the unit load method

Consider again our previous cantilever beam problem. Using the unit load method, we simply compute

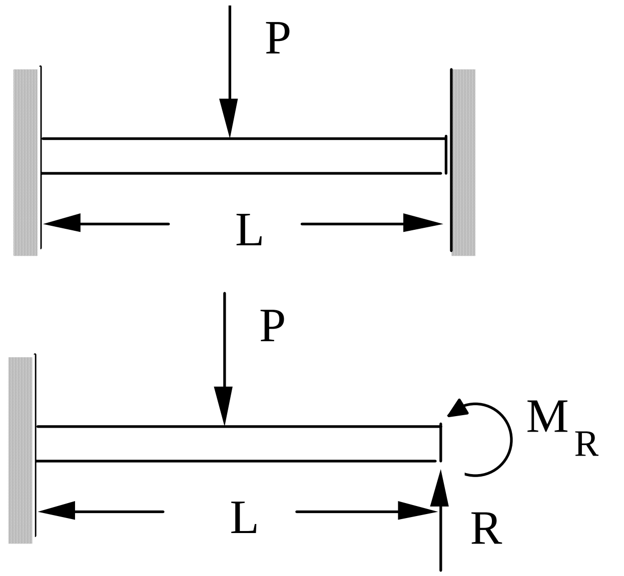
$$M(P,x) = -Px \quad (0 < x < L)$$
$$M_1(Q=1,x) = \begin{cases} 0 & (0 < x < L/2) \\ -(x - L/2) & (L/2 < x < L) \end{cases}$$

to arrive at Eq.(14) again, namely

$$\begin{aligned}\Delta_Q\Big|_{Q=0} &= \frac{1}{EI}\int_{L/2}^{L}(Px)(x - L/2)dx \\ &= \frac{5}{48}\frac{PL^3}{EI}\end{aligned}$$

Complimentary Strain Energy and the Principle of Least Work

All the previous uses of the principle of complimentary virtual work have been for statically determinant problems where we could directly write the complimentary strain energy in terms of the applied loads. For statically indeterminant problems this is not possible and at most we can do is write the strain energy in terms of the applied loads and a set of redundant forces and/or moments which arise from the over-constrained nature of statically indeterminant problems. For, example, consider a beam supported, as shown at its ends by two fixed supports.



It is obvious that this system is statically indeterminant to the second degree, i.e. there are two more (redundant) forces and moments than there are equations of equilibrium. If we take the right hand end reactions as these two redundants, then we could write the complimentary strain energy as $U^c = U^c(P, R, M_R)$.

Similarly, in a more general situation where we had a number of applied forces and moments (P_i, M_i) and a set of redundants (R_i, M_{Ri}) we could write $U^c = U^c(P_i, M_i, R_i, M_{Ri})$. If we imagine temporarily removing the support constraints that cause these redundants to exist, and let these redundants have virtual changes δR_i, keeping the external forces and moments constant and without violating equilibrium, then the principle of complimentary virtual work gives

$$\delta U^c = \sum_{i=1}^{n} \Delta_i \delta R_i + \sum_{i=1}^{m} \theta_i \delta M_{Ri} \tag{19}$$

where we have assumed there are *n* redundant forces and *m* redundant moments present. Thus, it follows, from Engesser's first theorem that

$$\frac{\partial U^c}{\partial R_i} = \Delta_i \quad (i = 1, \ldots, n)$$

$$\frac{\partial U^c}{\partial M_{Ri}} = \theta_i \quad (i = i, \ldots, m)$$

However, to obtain the solution to our original problem, we must impose the original constraints that generate the redundants, namely we must set

$$\Delta_i = 0 \quad (i = 1, \ldots, n)$$
$$\theta_i = 0 \quad (i = 1, \ldots, m)$$

which leads to *n*+*m* equations

$$\frac{\partial U^c}{\partial R_i} = 0 \quad (i = 1, \ldots, n)$$
$$\frac{\partial U^c}{\partial M_{Ri}} = 0 \quad (i = 1, \ldots, m) \tag{20}$$

that can be used to solve for the *n*+*m* unknown redundants.

From Eq.(20) it follows that the solution for a statically indeterminant problem satisfies

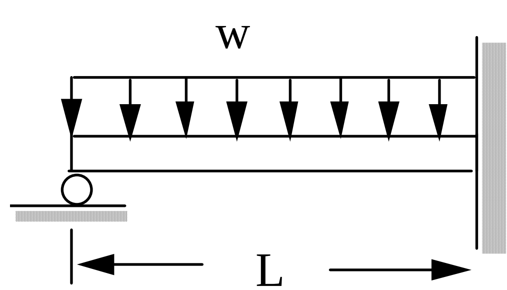
$$\delta U^c = \sum_{i=1}^{n} \frac{\partial U^c}{\partial R_i} \delta R_i + \sum_{i=1}^{m} \frac{\partial U^c}{\partial M_{Ri}} \delta M_{Ri} = 0 \tag{21}$$

which is called the *theorem of least work* (or *Engesser's second theorem*). Stated explicitly, this theorem says:

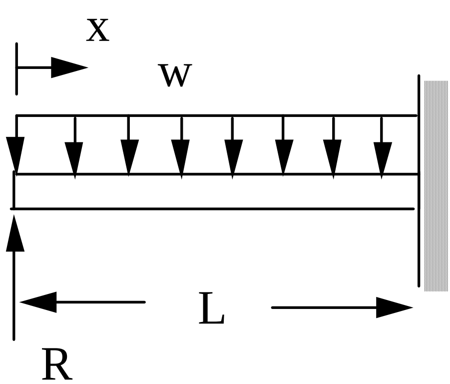
> Of all the possible values of the redundants that satisfy equilibrium for a statically indeterminant elastic system, the correct values of the redundants (those that satisfy both equilibrium and the given constraints) are those that make the complimentary strain energy stationary.

Example

For the statically indeterminant beam shown, determine the unknown reaction force at the left hand support



This problem is statically indeterminant to the first degree and we can take the desired reaction force *R* as the single redundant.



The bending moment in terms of the coordinate x shown is

$$M(x) = Rx - \frac{wx^2}{2}$$

so that

$$U^c(R) = U(R) = \frac{1}{2EI}\int_0^L \left(Rx - \frac{wx^2}{2} \right)^2 dx$$

and from

$$\frac{\partial U^c}{\partial R} = \Delta_R = \frac{1}{EI}\int_0^L \left(Rx - \frac{wx^2}{2} \right) x\, dx = 0$$

we find $R = \frac{3wL}{8}$.

Reciprocal Relations

A linear elastic body possesses some interesting properties that can be used to advantage in many cases in finding solutions to particular problems. One of these properties is that of *reciprocity*.

Consider a linear elastic body that occupies the volume Vand whose surface is S and let **x** be an arbitrary point in V and $\mathbf{x}_s$ an arbitrary point on S. Also, let $\sigma_{ij}^{(1)}(\mathbf{x}),\ e_{ij}^{(1)}(\mathbf{x}),\ u_j^{(1)}(\mathbf{x})$ be the stresses, strains, and displacements in V for this body due to a set of surface tractions, surface displacements, and body forces given by

$T_j^{(1)}(\mathbf{x}_s), u_j^{(1)}(\mathbf{x}_s), f_j^{(1)}(\mathbf{x}_s)$. Similarly, let $\sigma_{ij}^{(2)}(\mathbf{x}), e_{ij}^{(2)}(\mathbf{x}), u_j^{(2)}(\mathbf{x})$ be the stresses, strains and displacements due to $T_j^{(2)}(\mathbf{x}_s), u_j^{(2)}(\mathbf{x}_s), f_j^{(2)}(\mathbf{x}_s)$ for the same body. Then

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\int_V \sigma_{ij}^{(1)} e_{ij}^{(2)} dV = \sum_{i=1}^{3}\sum_{j=1}^{3}\int_V \sigma_{ij}^{(2)} e_{ij}^{(1)} dV$$

which follows directly from the fact that for both cases (1) and (2) the stress-strain relations give

$$\sigma_{ij}^{(1)} = \sum_{k=1}^{3}\sum_{l=1}^{3} C_{ijkl} e_{kl}^{(1)}$$
$$\sigma_{ij}^{(2)} = \sum_{k=1}^{3}\sum_{l=1}^{3} C_{ijkl} e_{kl}^{(2)}$$

However, because of the symmetry of the stresses we have for either case

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij} e_{ij} = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}\frac{\partial u_j}{\partial x_i}$$

so that

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\int_V \sigma_{ij}^{(1)} \frac{\partial u_j^{(2)}}{\partial x_i} dV = \sum_{i=1}^{3}\sum_{j=1}^{3}\int_V \sigma_{ij}^{(2)} \frac{\partial u_j^{(1)}}{\partial x_i} dV$$

From the use of the chain rule of calculus and the equations of equilibrium, for either case

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}\frac{\partial u_j}{\partial x_i} = \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial}{\partial x_i}\left(\sigma_{ij} u_j\right) + \sum_{j=1}^{3} f_j u_j$$

so that

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\int_V \frac{\partial}{\partial x_i}\left(\sigma_{ij}^{(1)} u_j^{(2)}\right) dV + \sum_{j=1}^{3}\int_V f_j^{(1)} u_j^{(2)} dV = \sum_{i=1}^{3}\sum_{j=1}^{3}\int_V \frac{\partial}{\partial x_i}\left(\sigma_{ij}^{(2)} u_j^{(1)}\right) dV + \sum_{j=1}^{3}\int_V f_j^{(2)} u_j^{(1)} dV \qquad (22)$$

But, by the divergence theorem, the first integrals on each side of Eq. (22) can be transformed to integrals over the surface of the body, i.e.

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\int_S \sigma_{ij}^{(1)} u_j^{(2)} n_i dS + \sum_{j=1}^{3}\int_V f_j^{(1)} u_j^{(2)} dV = \sum_{i=1}^{3}\sum_{j=1}^{3}\int_S \sigma_{ij}^{(2)} u_j^{(1)} n_i dS + \sum_{j=1}^{3}\int_V f_j^{(2)} u_j^{(1)} dV \qquad (23)$$

and for either case, the surface tractions are given by $T_j = \sum_{i=1}^{3}\sigma_{ij} n_i$,reducing Eq. (23) to

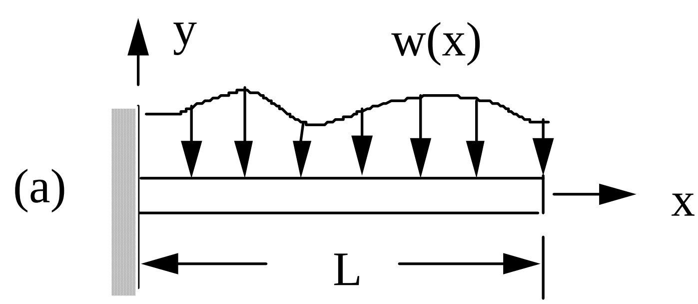
$$\sum_{j=1}^{3}\int_S T_j^{(1)} u_j^{(2)} dS + \sum_{j=1}^{3}\int_V f_j^{(1)} u_j^{(2)} dV = \sum_{j=1}^{3}\int_S T_j^{(2)} u_j^{(1)} dS + \sum_{j=1}^{3}\int_V f_j^{(2)} u_j^{(1)} dV \qquad (24)$$

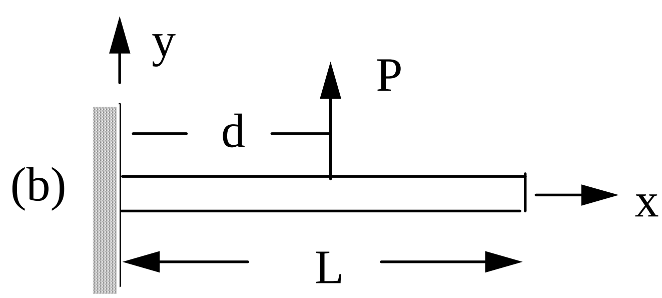
Eq. (24) is a statement of the *reciprocal theorem of Betti-Rayleigh*. Stated explicitly, this theorem says that the work done by the surface tractions and body forces of case (1) acting through the displacements of case (2) is equal to the work done by the surface tractions and body forces of case (2) acting through the displacements of case (1). For the case where all the forces (and moments) are concentrated, and the body forces are absent, the theorem can be written as

$$\sum_{i=1}^{N} P_i^{(1)} \Delta_i^{(2)} + \sum_{i=1}^{M} M_i^{(1)} \theta_i^{(2)} = \sum_{i=1}^{N} P_i^{(2)} \Delta_i^{(1)} + \sum_{i=1}^{M} M_i^{(2)} \theta_i^{(1)} \qquad (25)$$

Example use of reciprocity

Consider a cantilever beam acted upon by a distributed load $w(x)$ as shown in Fig. (a) below. We will call this problem case (1). Also, let the same beam be loaded by a concentrated load as shown in Fig. (b). we will call this problem case (2)





Then from the reciprocal theorem we have

$$-\int_0^L w(x) y^{(2)}(P,x,d)dx = P\,y^{(1)}(w,d)$$

where $y^{(2)}(P,x,d)$ is the displacement of the beam at *x* due to a load *P* acting at *d* and $y^{(1)}(w,d)$ is the displacement at *d* due to the distributed load *w(x).* The minus sign exists on the left side of the above equation because *w(x)* acts down. It is relatively easy to show that

$$y^{(2)}(P,x,d) = \begin{cases} \dfrac{Px^2}{6EI}(3d-x) & (0<x<d) \\ \dfrac{Pd^2}{6EI}(3x-d) & (d<x<L) \end{cases}$$

which gives

$$y^{(1)}(w,d) = \frac{-1}{6EI}\left\{\int_0^d w(x)x^2(3d-x)dx + \int_d^L w(x)d^2(3x-d)dx\right\} \tag{26}$$

Equation (26) gives the displacement of the beam at point *d* due to an arbitrary distributed load.