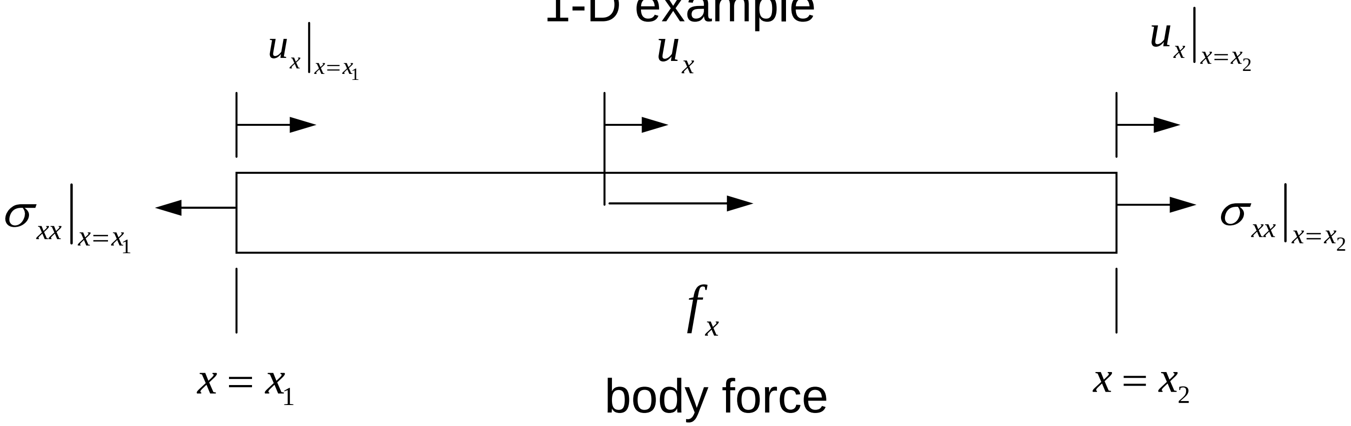
Principle of Virtual Work

1-D example



A = cross-sectional area

If we let the x-displacement change by a small "virtual" amount δu_x throughout the bar, those virtual displacement changes will do "virtual' work:

$$\delta W_v = \sigma_{xx} A \delta u_x\big|_{x=x_2} - \sigma_{xx} A \delta u_x\big|_{x=x_1} + \int_{x_1}^{x_2} f_x \delta u_x A dx$$

so

$$\delta W_v = \int_A \sigma_{xx} \delta u_x\big|_{x=x_1}^{x=x_2} dA + \int_{x_1}^{x_2} \int_A f_x \delta u_x \, dA \, dx$$

$$= \int_{x_1}^{x_2} \int_A \frac{d}{dx}\left(\sigma_{xx} \delta u_x\right) dA \, dx + \int_{x_1}^{x_2} \int_A f_x \delta u_x \, dA \, dx$$

$$\delta W_v = \int_{x_1}^{x_2}\int_A \cancel{\left(\frac{d\sigma_{xx}}{dx} + f_x\right)}\delta u_x\, dA\, dx + \int_{x_1}^{x_2}\int_A \sigma_{xx}\,\underset{\delta e_{xx}}{\underset{\parallel}{\delta\left(\frac{du_x}{dx}\right)}}\, dA\, dx$$

= 0 by equilibrium

virtual change of strain

$$\delta W_v = \int_V \sigma_{xx}\delta e_{xx}\, dV$$

But $\sigma_{xx} = \frac{du_o}{de_{xx}}$ for elastic material

so

$$\int_V \sigma_{xx}\delta e_{xx}\, dV = \int_V \frac{du_0}{de_{xx}}\delta e_{xx}\, dV$$

$$= \int_V \delta u_0\, dV = \delta\int_V u_0\, dV = \delta U$$

and we have

$$\delta W_v = \delta U$$

Thus, we have shown that if equilibrium is satisfied then the virtual work done by all the loads is equal to the virtual change in strain energy

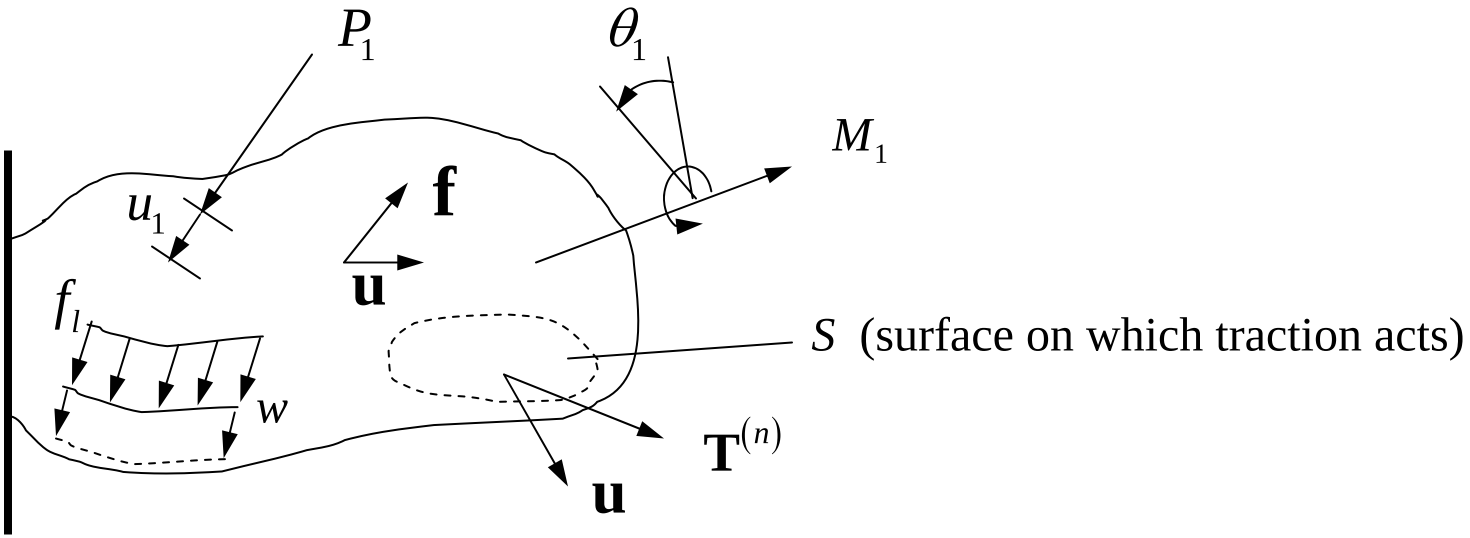
Equilibrium $\Longrightarrow$ $\delta W_v = \delta U$

But we can also show that if the principal of virtual work is satisfied for all possible virtual displacements then equilibrium will be satisfied

$\delta W_v = \delta U$

for all possible virtual displacements $\Longrightarrow$ Equilibrium

General 3-D Problems



P_1 … concentrated load u_1 … displacement at P_1 along it's direction

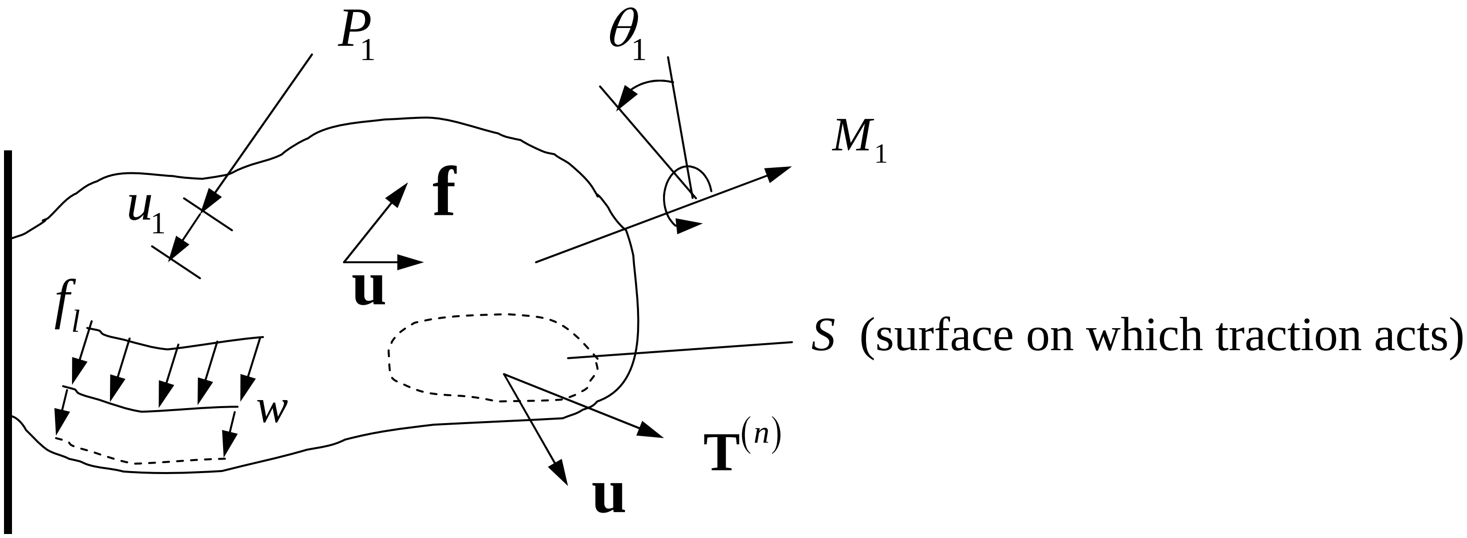
M_1 … concentrated couple θ_1 … angular rotation in a plane perpendicular to M_1 in its direction of rotation action

$\mathbf{f}$ … body force $\mathbf{u}$ … displacement

$\mathbf{T}^{(n)}$ … traction (stress) vector $\mathbf{u}$ … displacement

f_l … line force (force/unit length) w … displacement in direction of f_l

3-D Problems



$$\delta W_v = \int_S \mathbf{T}^{(n)} \cdot \delta \mathbf{u}\, dS + \int_V \mathbf{f} \cdot \delta \mathbf{u}\, dV + \int_l f_l \delta w ds + \underline{\sum_i P_i \delta u_i + \sum_i M_i \delta \theta_i}$$

for multiple loads and moments

As in the 1-D case can show that if equilibrium is satisfied

$$\delta W_v = \int_V \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma_{ij} \delta e_{ij}\ dV$$

$$\delta W_v = \int_V \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma_{ij} \delta e_{ij} \, dV$$

For elastic bodies where a strain energy exists

$$\sigma_{ij} = \frac{\partial u_0}{\partial e_{ij}}$$

$$\sum_{i=1}^{3} \sum_{j=1}^{3} \sigma_{ij} \delta e_{ij} = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial u_0}{\partial e_{ij}} \delta e_{ij} = \delta u_0$$

and, therefore

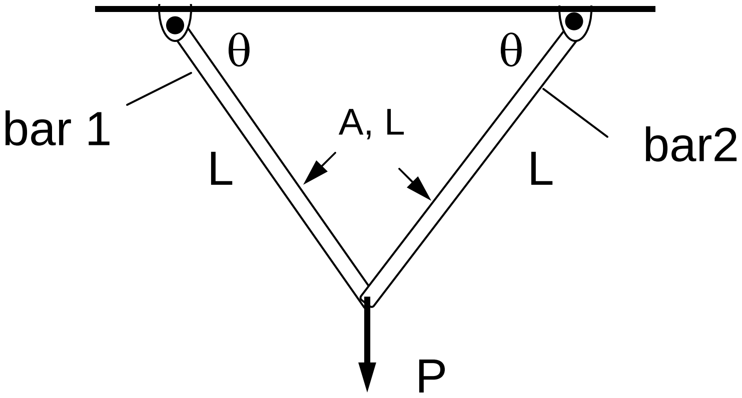
$$\delta W_v = \int_V \delta u_o \, dV = \delta \int_V u_0 dV = \delta U$$

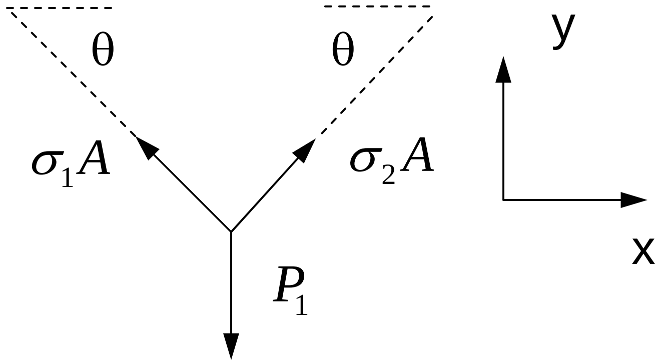
so in 3-D
problems also

$$\delta W_v = \delta U$$

Example:

Determine the stresses in bars 1 and 2





Equilibrium Approach

$$\overset{+}{\rightarrow}\sum F_x = 0$$

$$-\sigma_1 A\cos\theta + \sigma_2 A\cos\theta = 0$$

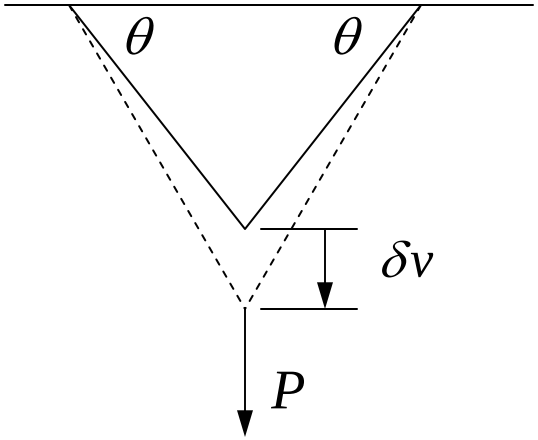
$$\sigma_1 = \sigma_2$$

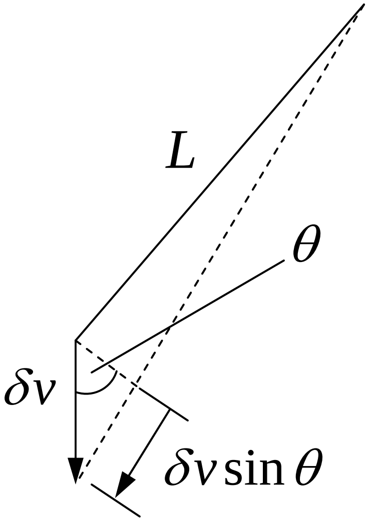
$$\overset{+}{\uparrow}\sum F_y = 0$$

$$\sigma_1 A\sin\theta + \sigma_2 A\sin\theta - P = 0$$

$$\sigma_1 = \sigma_2 = \frac{P}{2A\sin\theta}$$

Virtual Work Approach





virtual change of strain in
bar 2

$$\delta e_2 = \frac{\delta v \sin\theta}{L}$$

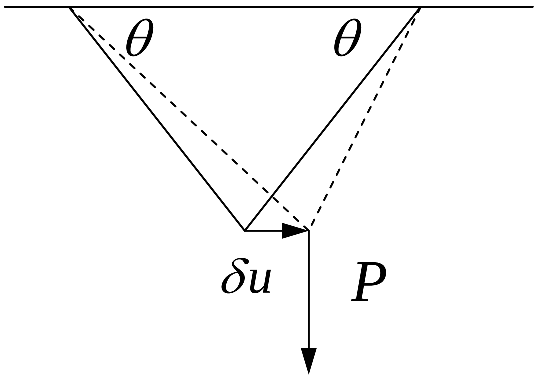
$$\delta e_1 = \delta e_2$$

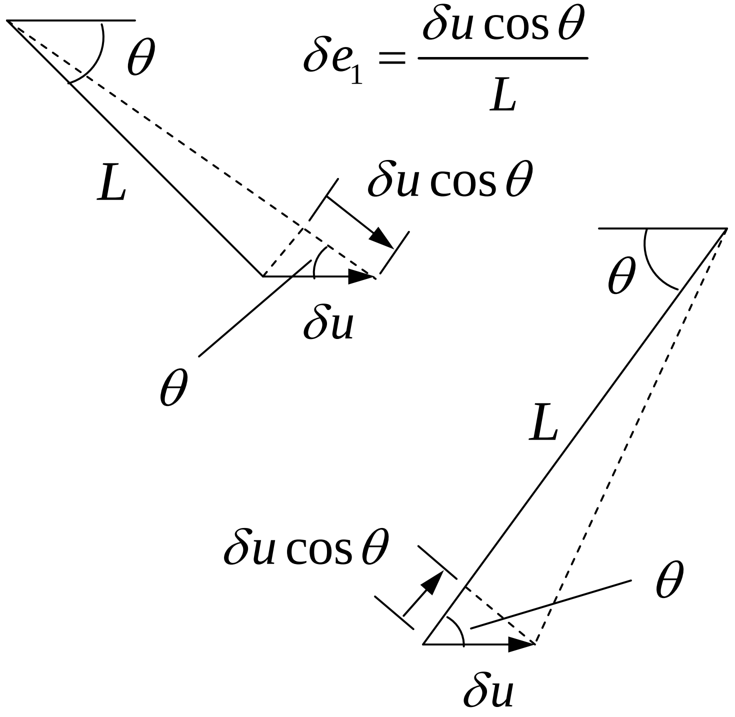
$$\delta W_v = \delta U$$

$$P\delta v = \int_{bar 1} \sigma_1 \delta e_1 dV + \int_{bar 2} \sigma_2 \delta e_2 dV$$

$$P\delta v = \sigma_1 \left(\frac{\delta v \sin\theta}{L} \right) AL + \sigma_2 \left(\frac{\delta v \sin\theta}{L} \right) AL$$

$$\left(\sigma_1 A \sin\theta + \sigma_2 A \sin\theta - P \right) \delta v = 0$$





$$\delta e_1 = \frac{\delta u\cos\theta}{L}$$

$$\delta e_2 = \frac{-\delta u\cos\theta}{L}$$

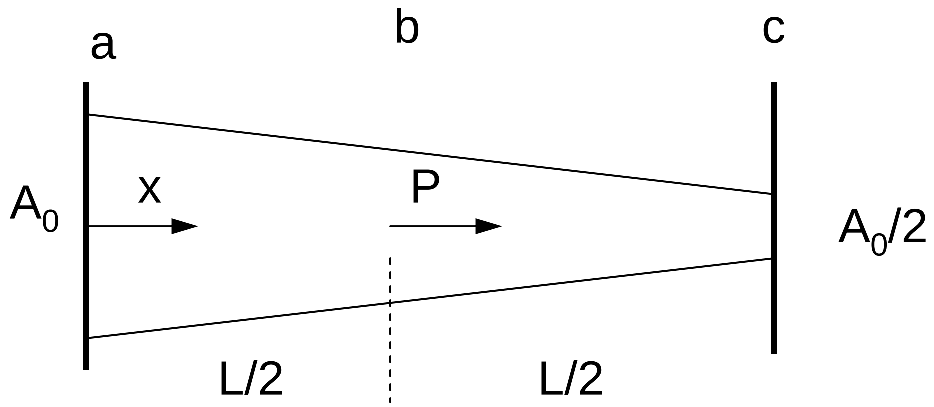
$$\delta W_v = \delta U$$

$$0 = \int_{bar\,1} \sigma_1 \delta e_1 dV + \int_{bar\,2} \sigma_2 \delta e_2 dV$$

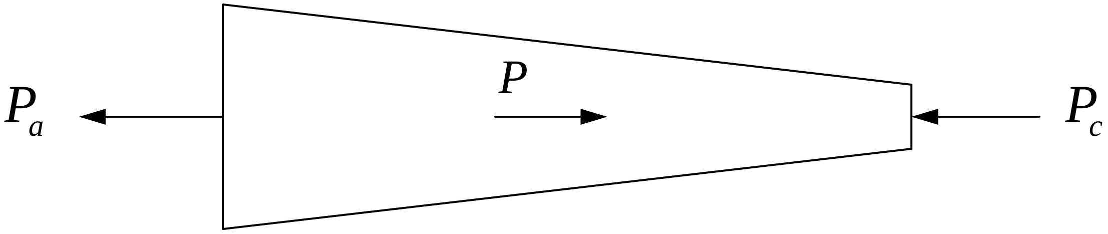
$$0 = \sigma_1\left(\frac{\delta u\cos\theta}{L}\right)AL + \sigma_2\left(\frac{-\delta u\cos\theta}{L}\right)AL$$

$$\left(\sigma_1 A\cos\theta - \sigma_2 A\cos\theta\right)\delta u = 0$$

Example 2: use of virtual work to obtain an approximate solution



Cross-sectional area $A(x) = A_0 - \frac{A_0 x}{2L}$



Problem: Determine P_a and P_c

This is a statically indeterminant problem. The exact solution is, as we will see

$$P_a = 0.585P$$
$$P_c = 0.415P$$

Note that the strains in a-b and b-c are not constants. They are given by:

$$e_{xx} = \frac{du_x}{dx} = \frac{\sigma_{xx}}{E}$$
$$= \begin{cases} \dfrac{P_a}{A(x)E} & in\ a-b \\ \dfrac{-P_c}{A(x)E} & in\ b-c \end{cases}$$

To obtain this exact solution we use equilibrium

$$P_a + P_c = P$$

together with the condition that the total displacement of the bar must be zero, i.e.

$$\int_0^L du_x = \int_0^L e_{xx} dx = 0$$

which gives

$$\frac{P_a}{EA_0} \int_0^{L/2} \frac{dx}{\left(1 - x/2L\right)} - \frac{P_c}{EA_0} \int_{L/2}^{L} \frac{dx}{\left(1 - x/2L\right)} = 0$$

Doing the integration gives

$$\frac{2P_c}{EA_0} \ln\left(\frac{1}{2}\right) - \frac{2\left(P_a + P_c\right)}{EA_0} \ln\left(\frac{3}{4}\right) = 0$$

so that exactly

$$P_c = \frac{\ln(3/4)}{\ln(1/2)} P \cong 0.415P$$

$$P_a = P - P_c \cong 0.585P$$

The displacement in the bar is given by

0 < x< L/2

$$\int_0^x du_x = u_x(x) - 0 = \frac{P_a}{EA_o} \int_0^x \frac{dx}{(1 - x/2L)}$$

$$= \frac{-2P_a L}{EA_0} \ln(1 - x/2L)$$

$$= \frac{-2PL}{EA_0} \left[1 - \frac{\ln(3/4)}{\ln(1/2)}\right] \ln(1 - x/2L)$$

and for L/2 < x < L

$$\int_L^x du_x = u_x(x) - 0 = \frac{-P_c}{EA_o}\int_L^x \frac{dx}{(1 - x/2L)}$$

$$= \frac{2P_cL}{EA_0}\left[\ln(1 - x/2L) - \ln(1/2)\right]$$

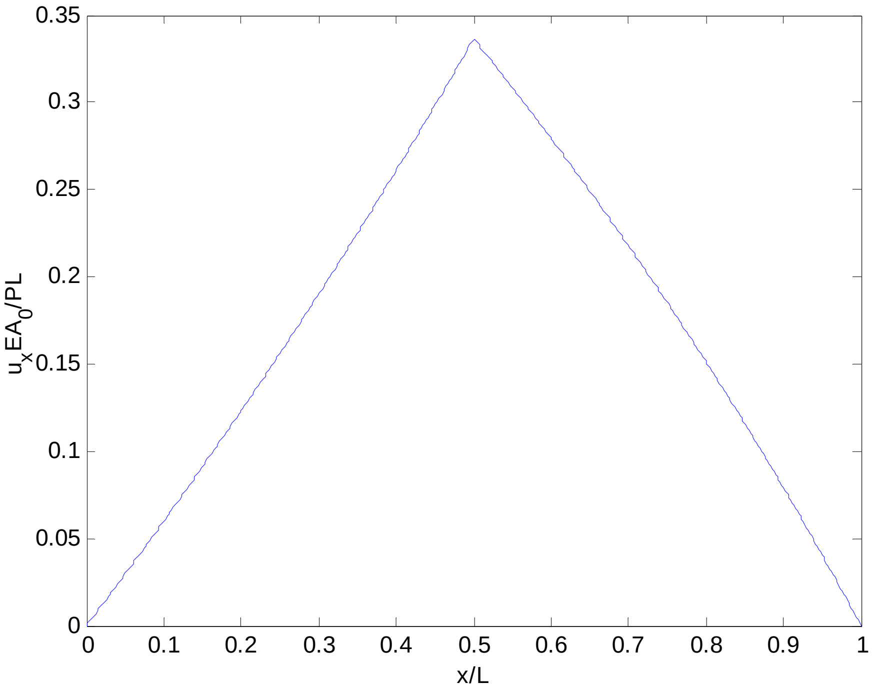
$$= \frac{2PL}{EA_0}\frac{\ln(3/4)}{\ln(1/2)}\left[\ln(1 - x/2L) - \ln(1/2)\right]$$

At x= L/2 we obtain

$$u_x \cong 0.3366\frac{PL}{EA_0}$$

A plot of the displacement in the bar gives

```
% script tapered_bar for calculating displacement of the tapered bar
% x here is the normalized distance x/L
% u is the normalized displacement u*E*Ao/(P*L)
%
x= linspace(0, 1, 500);
u = -2*(1-log(3/4)/log(1/2)).*log(1-x/2).*(x <= 0.5) + 2*(log(3/4)/log(1/2))* ...
    (log(1-x/2) - log(1/2)).*(x>0.5);
plot(x,u)
xlabel('x/L')
ylabel('u_xEA_0/PL')
```

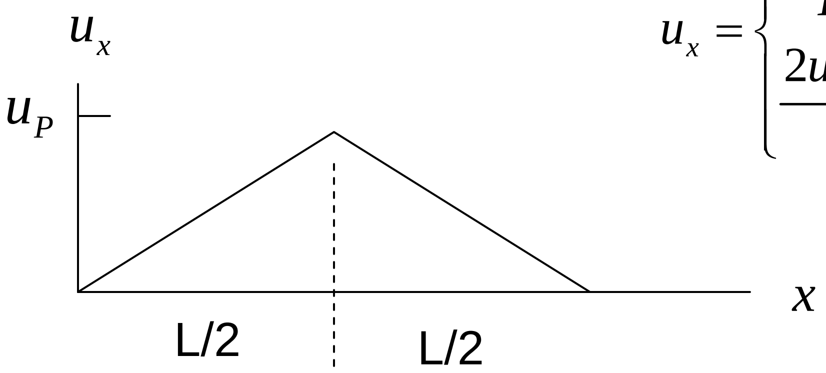


Virtual work approximate solution

Strain energy $$U = \int_0^{L/2} \frac{E\left[e_{xx}\left(x\right)\right]^2}{2} A\left(x\right)dx + \int_{L/2}^{L} \frac{E\left[e_{xx}\left(x\right)\right]^2}{2} A\left(x\right)dx$$

Now, approximate the displacement as:

$$u_x = \begin{cases} \dfrac{2u_P x}{L} & 0 < x < L/2 \\ \dfrac{2u_P\left(L-x\right)}{L} & L/2 < x < L \end{cases}$$



Then in a-b $$e_{xx} = \frac{du_x}{dx} = \frac{2u_P}{L}$$

and in b-c $$e_{xx} = \frac{du_x}{dx} = \frac{-2u_P}{L}$$

which are both constants

so $$U\left(u_P\right)=\frac{2Eu_P^2}{L^2}\int_0^{L/2}A\left(x\right)dx+\frac{2Eu_P^2}{L^2}\int_{L/2}^{L}A\left(x\right)dx$$

$$=\frac{2Eu_P^2A_0}{L^2}\int_0^{L/2}\left(1-\frac{x}{2L}\right)dx+\frac{2Eu_P^2A_0}{L^2}\int_{L/2}^{L}\left(1-\frac{x}{2L}\right)dx$$

$$=\frac{3}{2}\frac{Eu_P^2A_0}{L}$$

$$\delta W_v=\delta U$$

$$P\delta u_P=\frac{3Eu_PA_0}{L}\delta u_P$$

which gives $$u_P=\frac{PL}{3EA_0}\cong 0.3333\frac{PL}{EA_0}$$ which is close to the exact value

and the (constant) strains in a-b and b-c are

$$e_{ab}=\frac{2}{3}\frac{P}{EA_0},\quad e_{bc}=-\frac{2}{3}\frac{P}{EA_0}$$

Thus the end reactions are

$$P_a = Ee_{ab}A_0 = \frac{2}{3}P$$

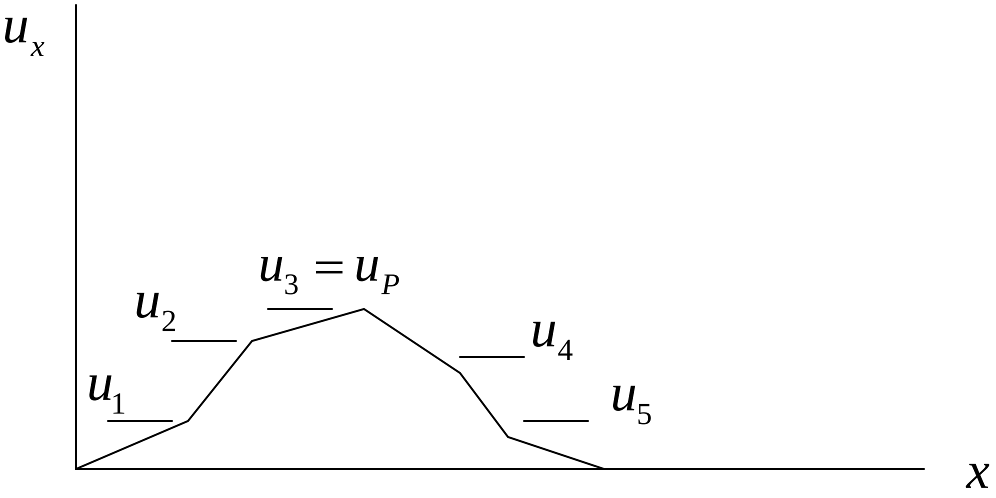
$$P_c = E\left|e_{bc}\right|\frac{A_0}{2} = \frac{1}{3}P$$

Compare these with the exact solution

$$P_a = 0.585P$$

$$P_c = 0.415P$$

This approximate solution is not bad considering the simplicity of the deformation we assumed. We could do much better by choosing a displacement that was able to follow more closely the exact result. For example, we could break the bar into small segments and write the displacement u_x in terms of the displacements at the ends of these segments:



If we express the strain energy in terms of those end displacements, i.e.

$$U = U\left(u_1, u_2, \ldots u_5\right)$$

Then from the principal of virtual work $\delta W_v = \delta U$

$$P\delta u_3 = \sum_{i=1}^{5} \frac{\partial U}{\partial u_i} \delta u_i$$

we obtain, by allowing each end displacement to vary independently

$$\frac{\partial U}{\partial u_1} = 0, \quad \frac{\partial U}{\partial u_2} = 0, \quad \frac{\partial U}{\partial u_3} = P$$

$$\frac{\partial U}{\partial u_4} = 0, \quad \frac{\partial U}{\partial u_5} = 0$$

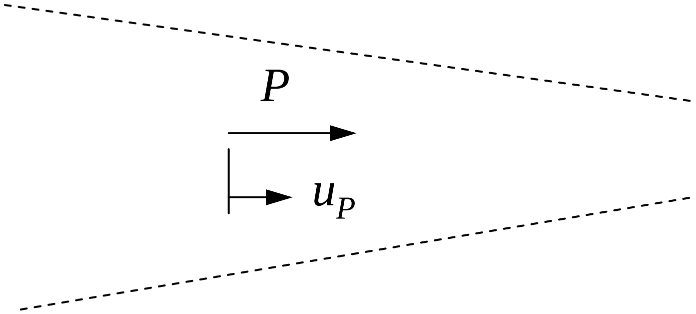
which for our problem would give five linear equations to solve for the five unknowns

$$u_1, u_2, u_3, u_4, u_5$$

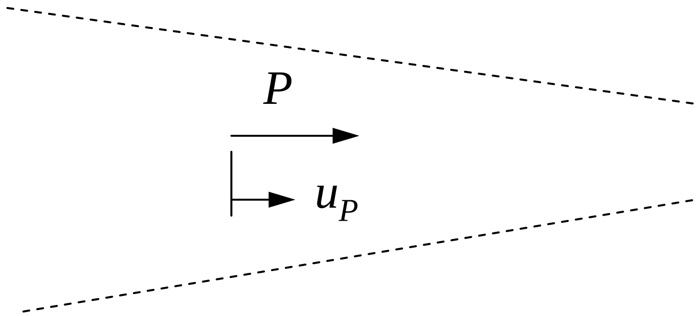
This is is essentially the approach used by ***Finite Elements*** to solve very complex 3-D stress analysis (and many other) problems

Note that the third of these equations gives

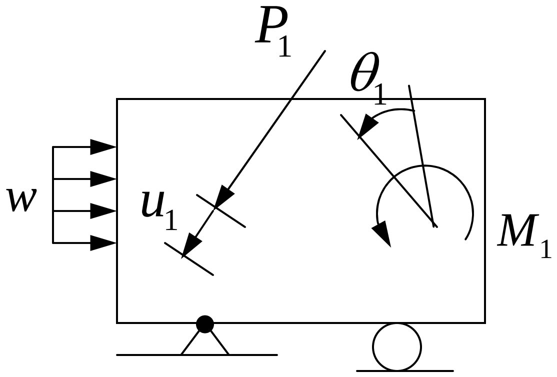
$$\frac{\partial U\left(u_1, \ldots, u_P, \ldots, u_5\right)}{\partial u_P} = P$$



$$\frac{\partial U\left(u_1,\dots,u_P,\dots,u_5\right)}{\partial u_P} = P$$

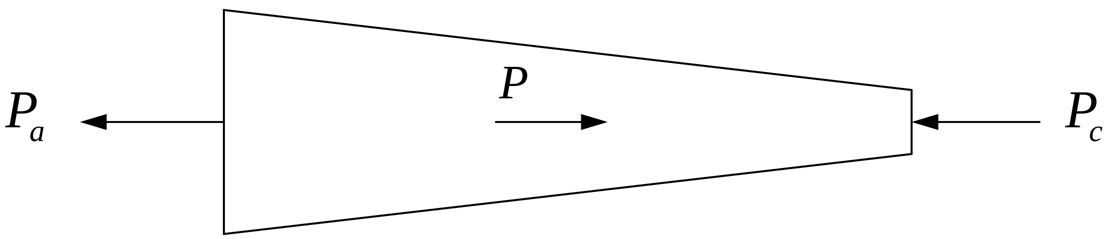


This is an example of <u>Castigliano's first theorem</u> which says that the derivative of the strain energy with respect to a displacement at a concentrated load in the direction of the load is equal to that load. a similar result holds for concentrated moments:



$$\frac{\partial U\left(u_1,\theta_1,\dots\right)}{\partial u_1} = P_1$$

$$\frac{\partial U\left(u_1,\theta_1,\dots\right)}{\partial \theta_1} = M_1$$



Also note that in applying the principle of virtual work, we used

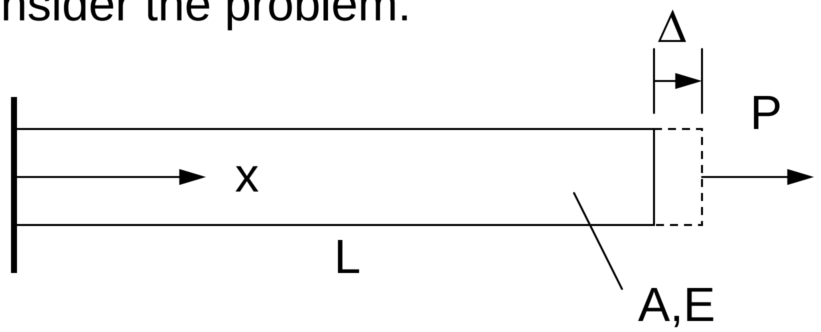
$$\delta U = \delta W_v = P\delta u_P$$

i.e. we only calculated the virtual work of only the external load and not the reactions. This was consistent with our choice of virtual changes of the displacement that vanished at the reactions and, hence those reactions did no work:

$$u_x = \begin{cases} \dfrac{2u_P x}{L} & 0 < x < L/2 \\ \dfrac{2u_P\left(L-x\right)}{L} & L/2 < x < L \end{cases} \quad \Longrightarrow \quad \delta u_x = \begin{cases} \dfrac{2\delta u_P x}{L} & 0 < x < L/2 \\ \dfrac{2\delta u_P\left(L-x\right)}{L} & L/2 < x < L \end{cases}$$

This is an example of choosing virtual displacement changes that satisfy the so-called essential boundary conditions on displacement.

However, there is nothing that prevents us from choosing virtual displacements that violate the essential boundary conditions. For example, consider the problem:



First, choose a displacement that satisfies the essential boundary condition $u_x\big|_{x=0} = 0$ $\quad u_x = \dfrac{\Delta x}{L}$

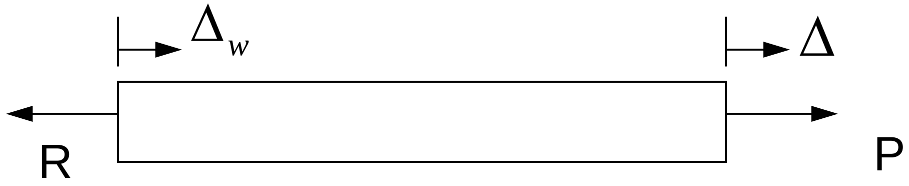
Then
$$U(\Delta) = \frac{1}{2}\int_0^L EA\left(\frac{du_x}{dx}\right)^2 dx$$
$$= \frac{1}{2}\frac{EA\Delta^2}{L}$$

and $\delta U = \dfrac{\partial U}{\partial \Delta}\delta\Delta = P\delta\Delta$ gives $\dfrac{EA\Delta}{L}\delta\Delta = P\delta\Delta$

or $P = \dfrac{EA\Delta}{L}$

Now, instead choose a displacement that violates the wall constraint

$$u_x = \Delta_w + \frac{(\Delta - \Delta_w)x}{L}$$



In this case

$$\frac{du_x}{dx} = \frac{\Delta - \Delta_w}{L}$$ and $$U = U(\Delta, \Delta_w) = \frac{1}{2}\frac{EA(\Delta - \Delta_w)^2}{L}$$

and the principle of virtual work gives, since now the reaction force, R, does work

$$\delta U = \frac{\partial U}{\partial \Delta}\delta\Delta + \frac{\partial U}{\partial \Delta_w}\delta\Delta_w = P\delta\Delta - R\delta\Delta_w$$

which gives $$\frac{EA(\Delta - \Delta_w)}{L}\delta\Delta + \frac{EA(\Delta_w - \Delta)}{L}\delta\Delta_w = P\delta\Delta - R\delta\Delta_w$$

In order to satisfy this equation for all displacement variations $\delta\Delta, \quad \delta\Delta_w$ we must have

$$P = \frac{EA(\Delta - \Delta_w)}{L}, \quad R = P$$

$$P = \frac{EA\left(\Delta - \Delta_w\right)}{L}, \quad R = P$$

These equations are just the exact solution for the bar with a elongation $\Delta - \Delta_w$

However, note that if we had tried to solve for the "nodal" displacements

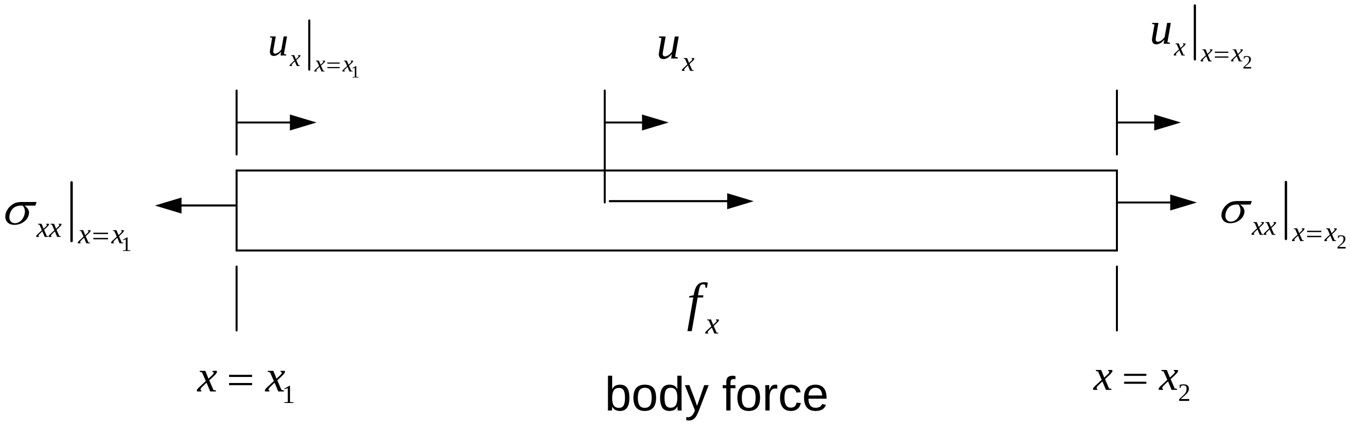
Δ, Δ_w we would obtain

$$\begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \begin{Bmatrix} \Delta \\ \Delta_w \end{Bmatrix} = \begin{Bmatrix} P \\ R \end{Bmatrix}$$

which is a singular system of equations. This occurs since obviously we cannot solve uniquely for Δ, Δ_w since we can always add a rigid body displacement $\Delta = \Delta_w = C$ and not affect the loads. This also occurs in Finite Elements where we must restrain the body so that rigid body displacements are eliminated or else we will end up with a singular system of equations for the "nodal" displacements.

Principle of Complimentary Virtual Work

1-D example



A = cross-sectional area

Now, let the displacements be held fixed and consider "virtual changes in the stresses and loads. Then the virtual complimentary work done by these changes is

$$\delta W_v^c = \int_A \delta\sigma_{xx} u_x \Big|_{x=x_1}^{x=x_2} dA + \int_0^L \int_A \delta f_x \, u_x \, dA \, dx$$

$$= \int_0^L \int_A \left[\frac{d}{dx}\left(\delta\sigma_{xx} u_x\right) + \delta f_x \, u_x \right] dA dx$$

or

$$\delta W_v^c = \int_0^L \int_A \left\{ \left[\cancel{\frac{d\left(\delta\sigma_{xx}\right)}{dx}} + \delta f_x \right] u_x + \delta\sigma_{xx} \underset{\underset{e_{xx}}{\shortparallel}}{\frac{du_x}{dx}} \right\} dA\,dx$$

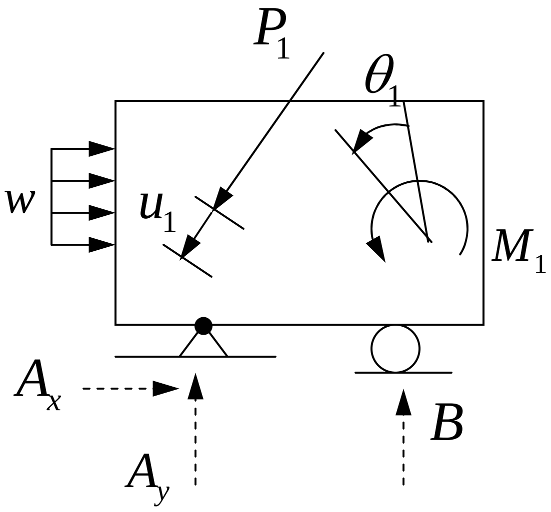
= 0 if virtual changes do
not violate equilibrium

$$\begin{aligned} \delta W_v^c &= \int_0^L \int_A \delta\sigma_{xx} e_{xx}\, dA\,dx \\ &= \int_V \delta u_0^c\, dV \\ &= \delta U^c \end{aligned}$$

virtual complimentary work = change in complimentary strain energy

$$\boxed{\delta W_v^c = \delta U^c}$$

Consider the problem where we have a combination of concentrated loads and moments plus other loadings on a body



If the problem is <u>statically determinant</u>, then we can use equilibrium and determine all the forces and moments. Then we can write the complimentary strain energy entirely in terms of the applied loads:

$$U^c = U^c\left(P_1, M_1, w, \ldots\right)$$

Consider now varying the applied load P_1 and moment M_1. Then from the principle of complimentary virtual work

$$\delta U^c = \frac{\partial U^c}{\partial P_1}\delta P_1 + \frac{\partial U^c}{\partial M_1}\delta\theta_1 = \delta P_1 u_1 + \delta M_1 \theta_1$$

But since these virtual changes are arbitrary, we must have

$$u_1 = \frac{\partial U^c}{\partial P_1}, \quad \theta_1 = \frac{\partial U^c}{\partial M_1}$$

Engesser's first theorem

If the body is linearly elastic $U^c = U$ and we have

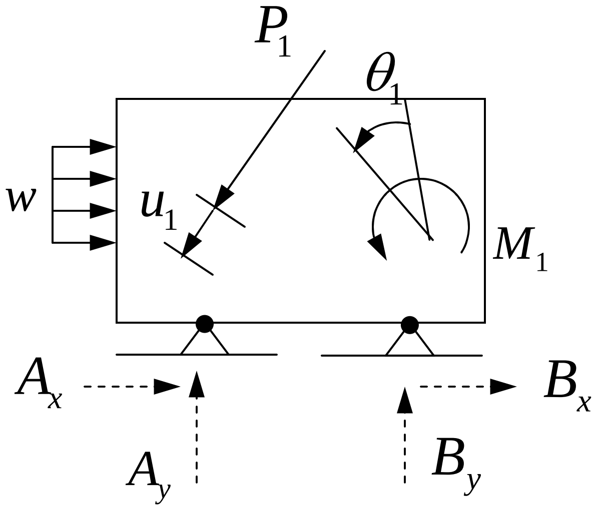
$$u_1 = \frac{\partial U}{\partial P_1}, \quad \theta_1 = \frac{\partial U}{\partial M_1}$$

Castigliano's second theorem

Note that the reactions A_x, A_y, B also vary when we vary P_1 and M_1 but these do no work since $(u_x)_A = (u_y)_A = (u_y)_B = 0$

Now, consider the case when the problem is statically indeterminant. In this case, we cannot use equilibrium to solve for all the reaction forces or moments. There will be some reactions left over which are unknowns. If we can vary those left over reactions independently without violating equilibrium, then they are called redundants.

For the problem shown below, for example, we have one redundant which we could take, for example, as B_y.



In this case we then could write

$$U^c = U^c\left(P_1, M_1, B_y, w, \ldots\right)$$

Now, imagine for the moment that we ignore the constraint $\left(u_y\right)_B = 0$ and allow B_y to vary. Then by the principle of complimentary virtual work we have

$$\delta U^c = \frac{\partial U^c}{\partial B_y}\delta B_y = \delta B_y\left(u_y\right)_B$$

which gives, since the redundant δB_y can be varied arbitrarily

$$\left(u_y\right)_B = \frac{\partial U^c}{\partial B_y}$$

If we now enforce the constraint $\left(u_y\right)_B = 0$ we obtain

$$\frac{\partial U^c\left(P_1, M_1, B_y, w, ...\right)}{\partial B_y} = 0$$

which is an equation we can use to solve for B_y

Note that the condition $$\frac{\partial U^{c}\left(P_1, M_1, B_y, w, \ldots\right)}{\partial B_y} = 0$$

also implies that $$\delta U^{c}\left(P_1, M_1, B_y, w, \ldots\right) = 0$$

which is a statement of Engesser's second theorem or the principle of least work:

Of all the possible values of the redundants $R_1, R_2, \ldots R_n$ that satisfy equilibrium for a statically indeterminant elastic system, the correct values of the redundants (those that satisfy both equilibrium and the given constraints) are those that make the complimentary strain energy stationary with respect to variations of those redundants.

$$\delta U^{c} = \sum_{m=1}^{n} \frac{\partial U^{c}}{\partial R_m} \delta R_m = 0$$