Torsion of Non-Circular Sections – The Prandtl Stress Function

In Saint Venant's theory of torsion for non-circular sections, the displacements are given by

$$\begin{aligned} u_z &= y\phi(x) \\ u_y &= -z\phi(x) \\ u_x &= \frac{d\phi}{dx}\psi(y,z) \end{aligned} \qquad (1)$$

If $d\phi/dx = \phi'$ is a constant, then the only non-zero stresses are

$$\begin{aligned} \sigma_{xz} &= G\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = G\left(\frac{\partial u_x}{\partial z} + \phi' y\right) \\ \sigma_{xy} &= G\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = G\left(\frac{\partial u_x}{\partial y} - \phi' z\right) \end{aligned} \qquad (2)$$

and all the equilibrium equations are satisfied if

$$\frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0 \qquad (3)$$

This equation can be satisfied automatically by writing the stresses in terms of a function, Φ, called the Prandtl stress function, where

$$\sigma_{xy} = \frac{\partial \Phi}{\partial z}, \quad \sigma_{xz} = -\frac{\partial \Phi}{\partial y} \qquad (4)$$

However, from Eq. (2), we have

$$\begin{aligned} G\frac{\partial u_x}{\partial z} &= -\frac{\partial \Phi}{\partial y} - G\phi' y \\ G\frac{\partial u_x}{\partial y} &= \frac{\partial \Phi}{\partial z} + G\phi' z \end{aligned}$$

which also implies

$$G\frac{\partial^2 u_x}{\partial z \partial y} = -\frac{\partial^2 \Phi}{\partial y^2} - G\phi'$$

$$G\frac{\partial^2 u_x}{\partial y \partial z} = \frac{\partial^2 \Phi}{\partial z^2} + G\phi'$$

However, these mixed derivatives of the displacement u_x must be equal, if we are to be able to integrate the strains to find this displacement, and this compatibility condition requires that the stress function satisfy

$$\frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = -2G\phi' \qquad (5)$$

or, equivalently,

$$\nabla^2 \Phi = -2G\phi' \qquad (6)$$

which is called Poisson's equation. We know that on the outer boundary of the bar we have no applied tractions so that

$$T_x^{(n)} = \sigma_{xy} n_y + \sigma_{xz} n_z = 0 \qquad (7)$$

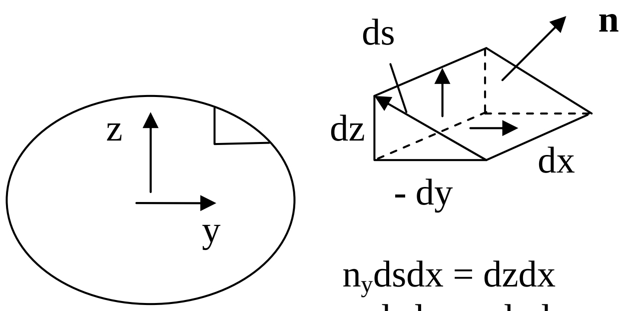
and y and z components of the traction vector are identically zero, so that in terms of the Prandtl stress function we have

$$\frac{\partial \Phi}{\partial z} n_y - \frac{\partial \Phi}{\partial z} n_z = 0 \qquad (8)$$

But by examining a small element near the surface (Fig.1), we see that Eq. (8) also implies that

$$\frac{\partial \Phi}{\partial z}\frac{dz}{ds} + \frac{\partial \Phi}{\partial y}\frac{dy}{ds} = 0 \qquad (9)$$

which says that Φ must be a constant on the boundary. For a cross-section with no holes, we can take the constant to be zero in general.



$n_y dsdx = dzdx$
$n_z dsdx = -dydx$

Fig. 1

To find the torque, T, in terms of the stress function, we start from the relation between the torque and stresses and write that relation in terms of the stress function:

$$
\begin{aligned}
T &= \int_A \left(\sigma_{xz} y - \sigma_{xy} z\right) dA \\
&= -\int_A \left(\frac{\partial \Phi}{\partial y} y + \frac{\partial \Phi}{\partial z} z\right) dA \\
&= -\int_A \left[\frac{\partial}{\partial y}(\Phi y) + \frac{\partial}{\partial z}(\Phi z) - 2\Phi\right] dA \\
&= -\oint_C \left[\Phi y\, n_y + \Phi z\, n_z\right] ds + 2\int_A \Phi dA \\
&= 2\int_A \Phi dA
\end{aligned}
$$

where the integral over the boundary C vanishes because $\Phi = 0$. Thus, the torque T is just twice the area under the $\Phi(y, z)$ surface. It can be shown that on the boundary C, the total shear stress takes on the largest value anywhere in the cross section. Since at the boundary this total shear stress must be tangent to the boundary (Fig. 2)

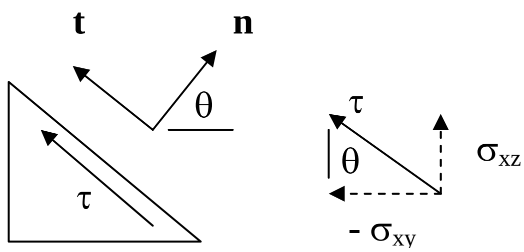


Fig. 2

we have

$$\sigma_{xz} = \tau\cos\theta$$
$$\sigma_{xy} = -\tau\sin\theta$$

However,

$$\frac{\partial\Phi}{\partial n} = \frac{\partial\Phi}{\partial y}n_y + \frac{\partial\Phi}{\partial z}n_z$$
$$= -\sigma_{xz}\cos\theta + \sigma_{xy}\sin\theta$$
$$= -\tau$$

so this maximum total shear stress is just the negative of the largest slope of the $\Phi(y,z)$ surface at the boundary.
We can make these results appear similar to the familiar formulas for the torsion of a circular cross section if we define a modified stress function $\Phi = G\phi'\overline{\Phi}$, where

$$\nabla^2\overline{\Phi} = -2 \; \textit{in the cross section}$$
$$\overline{\Phi} = 0 \; \textit{on the boundary}$$

If we let $T = GJ_{eff}\phi'$ then

$$J_{eff} = 2\int_A \overline{\Phi}dA$$

and for the maximum shearing stress we have

$$\tau_{\max} = \frac{T}{J_{eff}}\left(-\frac{\partial\overline{\Phi}}{\partial n}\right)_{\max\, \textit{on the boundary}}$$

Some Examples

1. Circular Cross Section of radius c

In this case we have

$$\overline{\Phi} = -\frac{1}{2}\left(y^2+z^2-c^2\right) = -\frac{1}{2}\left(r^2-c^2\right)$$
$$J_{eff} = 2\int_A \overline{\Phi}dA = 2\pi\int_{r=0}^{r=c}\left(c^2-r^2\right)rdr = \frac{\pi c^4}{2} = J$$
$$\tau_{\max} = \frac{T}{J_{eff}}\left(-\frac{\partial\overline{\Phi}}{\partial r}\right)_{r=c} = \frac{Tc}{J}$$

Note that in this case

$$\sigma_{xz} = -\frac{\partial \Phi}{\partial y} = G\phi' y = G\left(\frac{\partial u_x}{\partial z} + \phi' y\right)$$
$$\sigma_{xy} = \frac{\partial \Phi}{\partial z} = -G\phi' z = G\left(\frac{\partial u_x}{\partial y} - \phi' z\right)$$

so that

$$\frac{\partial u_x}{\partial y} = \frac{\partial u_x}{\partial z} = 0$$

which implies that u_x is at most a constant. We can take that constant as zero. Thus, there is no warping, as expected, in this case.

2. Elliptical Cross Section (with semi-major axes of lengths a and b along the y and z axes, respectively)

In this case

$$\overline{\Phi} = \frac{-a^2 b^2}{\left(a^2 + b^2\right)}\left(\frac{y^2}{a^2} + \frac{z^2}{b^2} - 1\right)$$
$$J_{eff} = \frac{\pi a^3 b^3}{\left(a^2 + b^2\right)}$$
$$\left(\tau_{\max}\right)_{z=\pm b} = \frac{2T}{\pi a b^2} \quad for\ b < a$$

If we integrate the shear stress expressions to find the warping displacement, u_x, we find

$$u_x = \frac{T\left(b^2 - a^2\right)}{\pi a^3 b^3 G} yz$$

3. Thin Rectangular Cross Section

Consider the case when then width of the rectangle, t, in the y direction is much less the length, b, in the z-direction. In this case we might expect that $\overline{\Phi} = f(y)$ so that

$$\nabla^2 \overline{\Phi} = \frac{d^2 f}{dy^2} = -2$$
$$f = 0 \quad on\ y = \pm t/2$$

which implies that

$$\overline{\Phi} = \left(\frac{t^2}{4} - y^2 \right)$$

$$J_{eff} = 2b \int_{y=-t/2}^{y=+t/2} \overline{\Phi} dy = \frac{1}{3} bt^3$$

$$\tau_{\max} = \frac{T}{J_{eff}} \left(\mp \frac{\partial \overline{\Phi}}{\partial y} \right)_{y=\pm t/2} = \frac{3T}{bt^2}$$

This value of J_{eff} for a thin rectangle can be compared with an approximate value that is good to several percent for a rectangle of arbitrary shape given by

$$J_{eff} = \frac{bt^3}{3} \left[1 - \frac{192}{\pi^5} \frac{t}{b} \tan\left(\frac{\pi b}{2t} \right) \right]$$

Consider a t/b ratio of 0.2, for example. In this case the above expression gives

$$J_{eff} = \frac{bt^3}{3} [1.017]$$

Recall, we mentioned that if we use the stresses directly to calculate J_{eff}, we get

$$\begin{aligned} T &= \int_A \left(\sigma_{xz} y - \sigma_{xy} z \right) dA \\ &= G\phi' \int_{y=-t/2}^{y=+t/2} 2y^2 b dy \\ &= G\phi' \frac{bt^3}{6} \end{aligned}$$

so that

$$J_{eff} = \frac{bt^3}{6}$$

which is a value that is only one half the value given above. The reason for this discrepancy is that we neglected the shear stresses σ_{xy} that develop near the ends of the cross section in this approximate solution. While those stresses are indeed small, they also have a larger moment arm (on the order of length b) than that of the shears stresses we did keep. To estimate the size of the contribution from the σ_{xy} stresses, we will assume that the stress function near the ends of the bar varies linearly from zero to its maximum value over a parabolically shaped region, A_{end}, of length h (see Fig. 3), i.e.

$$\overline{\Phi} \cong \frac{t^2}{4}\frac{z}{h}$$

so that

$$\sigma_{xy} \cong G\phi' \frac{t^2}{4h}$$

and we will estimate the torque produced by this stress (from both ends) as

$$\begin{aligned} T &\cong 2\left[\sigma_{xy}\frac{b}{2}\int_{A_{end}} dA\right] = G\phi' \frac{t^2}{4h} b\left(\frac{2th}{3}\right) \\ &= G\phi' \frac{bt^3}{6} \end{aligned}$$

which is equal to the torque contributions calculated from σ_{xz} so that the total torque from both stresses gives the correct result.

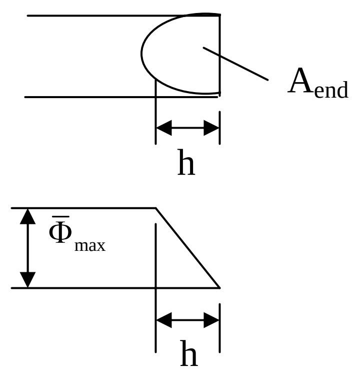


Fig. 3

To evaluate the warping function for the thin rectangle, note that for our approximate solution

$$\sigma_{xy} = \frac{\partial \Phi}{\partial z} = G\left(\frac{\partial u_x}{\partial y} - \phi' z\right) = 0$$

$$\sigma_{xz} = -\frac{\partial \Phi}{\partial y} = G\left(\frac{\partial u_x}{\partial z} + \phi' y\right) = 2G\phi' y$$

so that

$$\frac{\partial u_x}{\partial y} = \phi' z \, , \frac{\partial u_x}{\partial z} = \phi' y$$

Integrating both of these equations and comparing, we find

$$u_x = \phi' yz + f(z)$$
$$u_x = \phi' yz + g(y)$$

with *f* and *g* both arbitrary functions. Equating these two expressions, we find that we must have *f* = *g* = a constant, where we can take this constant equal to zero.

The Membrane Analogy

One of the reasons the Prandtl stress function approach has been a popular way to analyze torsion problems is that there is a correspondence between the behavior of this stress function and the deflection of a thin membrane under pressure.

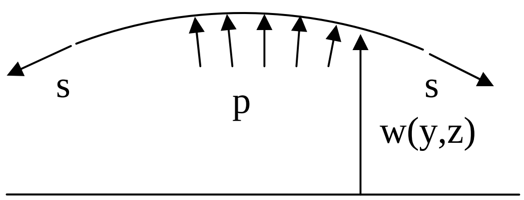


Fig.4

For example, Fig. 4 shows a side view of an inflated membrane under an internal pressure, p, where w is the vertical deflection of the membrane and s is the membrane tension. From equilibrium of a membrane element, one can show that the membrane satisfies

$$\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = -\frac{p}{s}$$
$$w = 0 \quad \textit{on the boundary}$$

which is identical to the solution of the torsion problem (for a simply connected region) in terms of the normalized Prandtl stress function if we make the substitution

$$w = \frac{p}{2s}\overline{\Phi}$$

Torsion of Multiply- Connected Cross Sections

We showed previously that the stress function had to be a constant on any unloaded boundary of the cross section. This is true for all boundaries of a multiply-connected cross section, including the holes. When we have a simply connected cross section we can always take the single constant on that boundary to be zero since we can always add or subtract a constant from the stress function without affecting the stresses or strains. However, if there are multiple holes in the cross section, the other constants are in

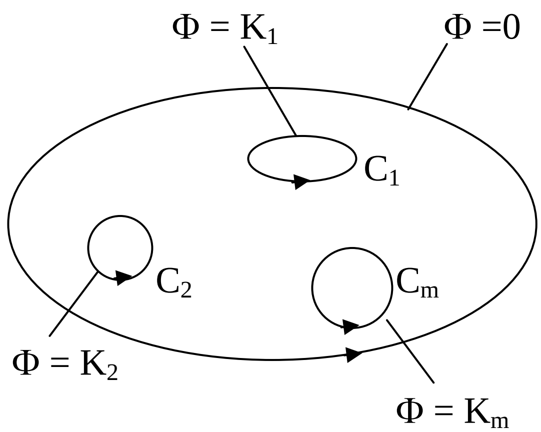


Fig. 5

general not zero as shown in Fig. 5.

To find these *m* constants we need to specify *m* additional conditions on our problem. These conditions can be obtained by requiring that the u_x displacement must be single-valued. This single-valuedness can be assured if we require that

$$\oint_{C_j} du_x = 0 \quad (j = 1, 2, \ldots m)$$

But, recall

$$\begin{aligned} u_x &= \phi' \psi(y,z) \quad , \phi' = d\phi / dx = constant \\ u_y &= -z\phi(x) \\ u_z &= y\phi(x) \end{aligned}$$

and

$$\sigma_{xy} = G\left(\frac{\partial u_x}{\partial y} - z\phi'\right)$$
$$\sigma_{xz} = G\left(\frac{\partial u_x}{\partial z} + y\phi'\right)$$

so that placing these relations into

$$\oint_{C_j} du_x = \oint_{C_j} \left(\frac{\partial u_x}{\partial y} dy + \frac{\partial u_x}{\partial z} dz\right) = 0$$

we obtain

$$\oint_{C_j} (\sigma_{xy} dy + \sigma_{xz} dz) + G\phi' \oint_{C_j} (zdy - ydz) = 0$$

where the integrals are in a counterclockwise sense. We can write these in terms of the unit normal (see Fig.1) components and the Prandtl stress function as

$$-\oint_{C_j} \left(\frac{\partial \Phi}{\partial z} n_z + \frac{\partial \Phi}{\partial y} n_y\right) ds - G\phi' \oint_{C_j} (zn_z + yn_y) ds = 0$$

or, equivalently, using Gauss' theorem

$$\begin{aligned}\oint_{C_j} \left(-\frac{\partial \Phi}{\partial n}\right) ds &= G\phi' \int_{A_j} \left[\frac{\partial}{\partial z}(z) + \frac{\partial}{\partial y}(y)\right] dA \\ &= 2G\phi' \int_{A_j} dA \\ &= 2G\phi' A_j\end{aligned}$$

where A_j is the area that the contour C_j encloses. Thus, to find the *m* constants we have to satisfy the m conditions

$$\oint_{C_j} \left(-\frac{\partial \Phi}{\partial n}\right) ds = 2G\phi' A_j$$

or, equivalently, in terms of the total shear stress on these boundaries

$$\oint_{C_j} \tau \, ds = 2G\phi' A_j$$