

Torsion of Bars with Circular Cross Sections –A Review

The twisting of a bar of a circular cross section by an axial torque is normally considered in strength of materials courses, as shown in Fig. 1. This problem can be analyzed by assuming that each planar cross section of the bar remains planar and simply rotates about the bar axis by a small amount, $\phi(x)$, producing a total twist $\phi_0 = \phi(L)$ in the bar at the end where the torque, T_0 , is applied.

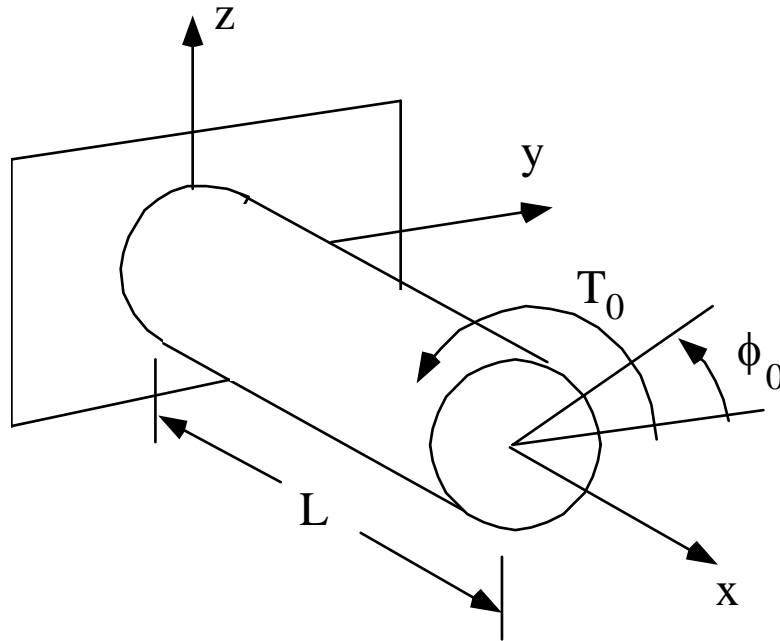


Fig. 1

Under this assumption, the displacements of each cross section are given by

$$\begin{aligned} u_x &= u_r = 0 \\ u_\theta &= r\phi(x) \end{aligned} \tag{1}$$

as shown in Fig. 2

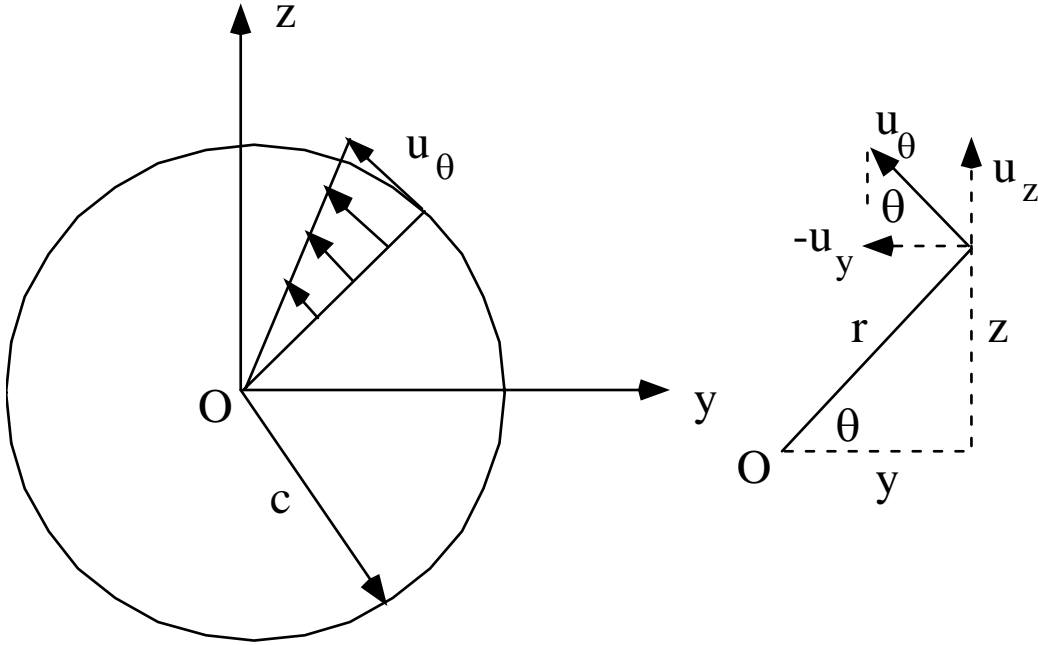


Fig. 2

Where r is the radial distance from the center of the bar. In terms of y - and z -components of the displacement then the displacements of the bar are

$$\begin{aligned}
 u_x &= 0 \\
 u_y &= -u_\theta \sin \theta = -r\phi(x) \sin \theta \\
 &= -z\phi(x) \\
 u_z &= u_\theta \cos \theta = r\phi(x) \cos \theta \\
 &= y\phi(x)
 \end{aligned} \tag{2}$$

These displacements produce strains in the bar given by

$$\begin{aligned}
 e_{xx} &= \frac{\partial u_x}{\partial x} = 0, e_{yy} = \frac{\partial u_y}{\partial y} = 0, e_{zz} = \frac{\partial u_z}{\partial z} = 0 \\
 \gamma_{xy} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = -\frac{d\phi}{dx} z \\
 \gamma_{xz} &= \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = \frac{d\phi}{dx} y \\
 \gamma_{yz} &= \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} = 0
 \end{aligned} \tag{3}$$

so that by generalized Hooke's law the only nonzero stresses in the bar are the shear stresses

$$\begin{aligned}\sigma_{xy} &= G\gamma_{xy} = -G\phi'z \\ \sigma_{xz} &= G\gamma_{xz} = G\phi'y\end{aligned}\tag{4}$$

where $\phi' = \frac{d\phi}{dx}$ is the twist per unit length.

For the problem of Fig. 1, the internal torque T is a constant (since $T = T_0$ throughout the bar by equilibrium) so as we will see ϕ' is also a constant. Thus, when the stresses of Eq. (4) are placed into the local equilibrium equations, those equations are satisfied identically. If we compute the moment that the shear stresses of Eq. (4) produce about the axis of the bar and equate that moment to the total internal torque of the cross section, then we find (see Fig. 3)

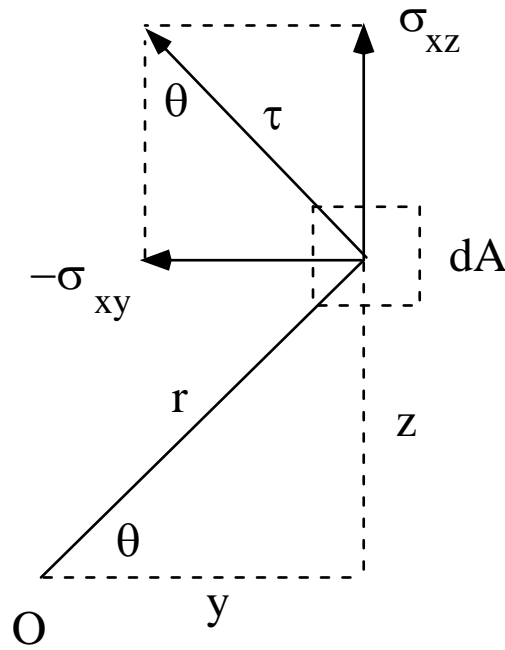


Fig. 3

$$\begin{aligned}
T &= \int_A (y\sigma_{xz} - z\sigma_{xy}) dA \\
&= G\phi' \int_A (y^2 + z^2) dA \\
&= G\phi' J
\end{aligned} \tag{5}$$

where J is given by

$$J = \int_A (y^2 + z^2) dA = \int_A r^2 dA \tag{6}$$

i.e. it is the polar moment of cross sectional area A. Eq.(5) shows explicitly that since T is a constant, ϕ' must also be a constant, as stated previously. The shear stress then can also be written as

$$\begin{aligned}
\sigma_{xy} &= \frac{-Tz}{J} \\
\sigma_{xz} &= \frac{Ty}{J}
\end{aligned} \tag{7}$$

and the total shear stress, τ , is given by (see Fig. 3)

$$\begin{aligned}
\tau &= -\sigma_{xy} \sin\theta + \sigma_{xz} \cos\theta \\
&= -\sigma_{xy} \frac{z}{r} + \sigma_{xz} \frac{y}{r} \\
&= \frac{Tr}{J}
\end{aligned} \tag{8}$$

which a familiar strength of materials result and shows that the maximum shear stress occurs on the outer boundary where $r = c$.

Summary

For a circular cross section, the displacements are assumed to be due to a pure rotation (twisting) of each cross section with no displacement in the axial direction, so

$$\begin{aligned}
u_x &= 0 \\
u_y &= -z\phi(x) \\
u_z &= y\phi(x)
\end{aligned}$$

If the twist per unit length is a constant, this assumption on the deformation means that the only non-zero stresses are given by

$$\sigma_{xy} = -G\phi'z$$

$$\sigma_{xz} = G\phi'y$$

These stresses can be used to relate the internal torque, T , to the twist per unit length and to find the maximum shear stress in the bar. These relations are given by

$$T = G\phi'J$$

$$\tau_{\max} = \frac{Tc}{J}$$

where

$$J = \int_A (y^2 + z^2) dA$$

is the polar area moment of the cross section.

Torsion of Bars with Non-Circular Cross Sections

In the last section it was assumed that each planar cross section of a circular bar remains planar and merely rotates (twists) about the central axis of the bar. When a torque is applied to a non-circular cross section, the cross section again rotates about the central axis but there is also a significant distortion of the cross section in x -direction, i.e. the cross sections both twist and warp.

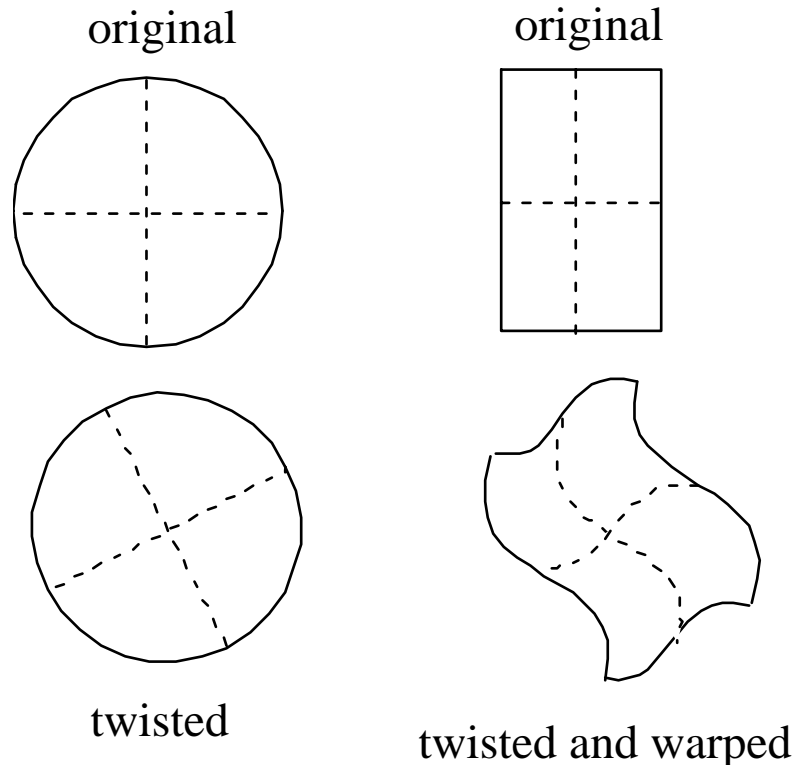


Fig. 4

We can incorporate this warping action into our previous description of the displacements of the bar if we replace Eq. (2) by

$$\begin{aligned} u_x &= \phi' \psi(y, z) \\ u_y &= -z\phi(x) \\ u_z &= y\phi(x) \end{aligned} \quad (9)$$

where ψ represents the out-of-plane warping (that has for convenience been multiplied by the constant twist per unit length, ϕ'). Then the strains are given by

$$\begin{aligned} e_{xx} &= \frac{\partial u_x}{\partial x} = 0, e_{yy} = \frac{\partial u_y}{\partial y} = 0, e_{zz} = \frac{\partial u_z}{\partial z} = 0 \\ \gamma_{xy} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = \phi' \left(\frac{\partial \psi}{\partial y} - z \right) \\ \gamma_{xz} &= \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = \phi' \left(\frac{\partial \psi}{\partial z} + y \right) \\ \gamma_{yz} &= \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} = \phi(x) - \phi(x) = 0 \end{aligned} \quad (10)$$

so that the only non-zero stresses again are

$$\begin{aligned} \sigma_{xy} &= G\phi' \left(\frac{\partial \psi}{\partial y} - z \right) \\ \sigma_{xz} &= G\phi' \left(\frac{\partial \psi}{\partial z} + y \right) \end{aligned} \quad (11)$$

These stresses must satisfy the equilibrium equations, which are

$$\begin{aligned} \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} &= 0 \\ \frac{\partial \sigma_{xy}}{\partial x} &= 0 \\ \frac{\partial \sigma_{xz}}{\partial x} &= 0 \end{aligned} \quad (12)$$

The second and third equilibrium equations are satisfied identically, while the first equation requires

$$\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 0 \quad (13)$$

so that the warping function must satisfy Laplace's equation. The boundary conditions that also must be satisfied by this warping function can be obtained by examining the stresses on the boundary.

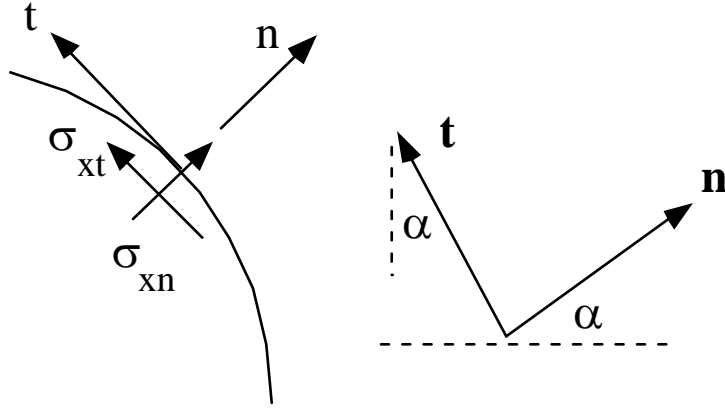


Fig. 5

From Fig. 5 we see that the shear stress σ_{xn} must vanish on the boundary since there is no corresponding component σ_{nx} acting on the lateral ($n = \text{constant}$) surface of the bar in the x -direction. Since

$$\begin{aligned} \sigma_{xn} &= \sigma_{xy} \cos \alpha + \sigma_{xz} \sin \alpha \\ &= \sigma_{xy} n_y + \sigma_{xz} n_z \\ \sigma_{xt} &= -\sigma_{xy} \sin \alpha + \sigma_{xz} \cos \alpha \\ &= -\sigma_{xy} n_z + \sigma_{xz} n_y \end{aligned} \quad (14)$$

from $\sigma_{xn} = 0$, from Eqs. (14) and 11) we find

$$\frac{\partial \psi}{\partial y} n_y + \frac{\partial \psi}{\partial z} n_z - zn_y + yn_z = 0 \quad (15)$$

or, equivalently

$$\frac{\partial \psi}{\partial n} = zn_y - yn_z \quad (16)$$

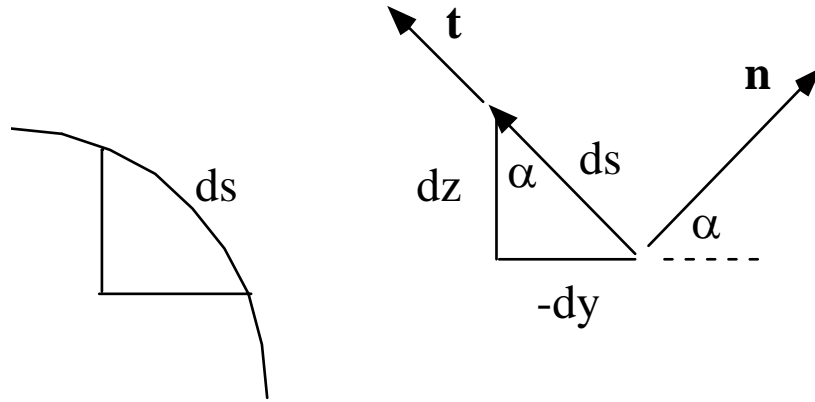


Fig. 6

However, if we let s be the arc length along the boundary (in the t -direction), we have

$$\begin{aligned}
 n_y &= \cos \alpha = \frac{dz}{ds} \\
 n_z &= \sin \alpha = -\frac{dy}{ds}
 \end{aligned}
 \tag{17}$$

so Eq. (16) becomes, finally

$$\begin{aligned}
 \frac{\partial \psi}{\partial n} &= z \frac{dz}{ds} + y \frac{dy}{ds} \\
 &= \frac{1}{2} \frac{d}{ds} (y^2 + z^2)
 \end{aligned}
 \tag{18}$$

which is the boundary condition that the warping function must satisfy. Note that one solution of Eq. (13) subject to the boundary condition of Eq. (18) is a warping function that is an arbitrary constant. However, this constant can be taken as zero since it only corresponds to a pure translation of the bar in the x -direction.

Once the warping function is obtained by solving Laplace's equation subject to the boundary conditions of Eq. (18), we can also obtain the relationship between the torque and the twist per unit length. To do this recall,

$$\begin{aligned}
 T &= \int_A (y\sigma_{xz} - z\sigma_{xy}) dA \\
 \sigma_{xy} &= G\phi' \left(\frac{\partial \psi}{\partial y} - z \right) \\
 \sigma_{xz} &= G\phi' \left(\frac{\partial \psi}{\partial z} + y \right)
 \end{aligned} \tag{19}$$

so that

$$\begin{aligned}
 T &= G\phi' \int_A \left[(y^2 + z^2) + \left(y \frac{\partial \psi}{\partial z} - z \frac{\partial \psi}{\partial y} \right) \right] dA \\
 &= G\phi' J_{eff}
 \end{aligned} \tag{20}$$

where J_{eff} is the effective polar area moment for the cross section. This effective area moment can be expressed in several different equivalent forms. Another form is

$$J_{eff} = \int_A \left[\left(y + \frac{\partial \psi}{\partial z} \right)^2 + \left(z - \frac{\partial \psi}{\partial y} \right)^2 \right] dA \tag{21}$$

It can be shown that the largest shear stress, τ_{max} , occurs at some point of the outer boundary of the cross section. Since on the boundary the shear stress is tangent to the boundary we have

$$\begin{aligned}
 \tau_{max} &= \{ \sigma_{xt} \}_{max} = \left\{ -\sigma_{xy} n_z + \sigma_{xz} n_y \right\}_{max} \\
 &= G\phi' \left\{ \left(z - \frac{\partial \psi}{\partial y} \right) n_z + \left(y + \frac{\partial \psi}{\partial z} \right) n_y \right\}_{max}
 \end{aligned} \tag{22}$$

so that if we also define an effective maximum radius, c_{eff} , as

$$c_{eff} = \left\{ \left(z - \frac{\partial \psi}{\partial y} \right) n_z + \left(y + \frac{\partial \psi}{\partial z} \right) n_y \right\}_{max} \tag{23}$$

then from Eqs. (20), (22) the maximum shear stress can be expressed in a form entirely similar to that of the circular cross section, namely

$$\tau_{\max} = \frac{Tc_{\text{eff}}}{J_{\text{eff}}} \quad (24)$$

Summary

For the torsion of a non-circular cross section we assumed the cross section both twist and warp so that the displacements are given by

$$\begin{aligned} u_x &= \phi' \psi(y, z) \\ u_y &= -z\phi(x) \\ u_z &= y\phi(x) \end{aligned} \quad (25a)$$

where ψ is the warping function. If the twist per unit length is a constant, this assumption on the displacement led to the only non-zero stresses being

$$\begin{aligned} \sigma_{xy} &= G\phi' \left(\frac{\partial \psi}{\partial y} - z \right) \\ \sigma_{xz} &= G\phi' \left(\frac{\partial \psi}{\partial z} + y \right) \end{aligned} \quad (25b)$$

We also showed that the equilibrium equations and boundary conditions for these stresses are satisfied if the warping function satisfies Laplace's equation

$$\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 0 \quad (25c)$$

subject to the conditions on the edge, C , of the cross section given by

$$\frac{\partial \psi}{\partial n} = \frac{1}{2} \frac{d}{ds} (y^2 + z^2) \quad (25d)$$

Then we found that we could obtain the internal torque and maximum shear stress in the bar in a form similar to the circular bar case as

$$\begin{aligned} T &= G\phi' J_{\text{eff}} \\ \tau_{\max} &= \frac{Tc_{\text{eff}}}{J_{\text{eff}}} \end{aligned} \quad (25e)$$

where

$$J_{eff} = \int_A \left[\left(z - \frac{\partial \psi}{\partial y} \right)^2 + \left(y + \frac{\partial \psi}{\partial z} \right)^2 \right] dA \quad (25f)$$

$$c_{eff} = \left\{ \left(z - \frac{\partial \psi}{\partial y} \right) n_z + \left(y + \frac{\partial \psi}{\partial z} \right) n_y \right\}_{max\ on\ C}$$

Example

Consider the torsion of a very thin rectangular section as shown in Fig. 7.

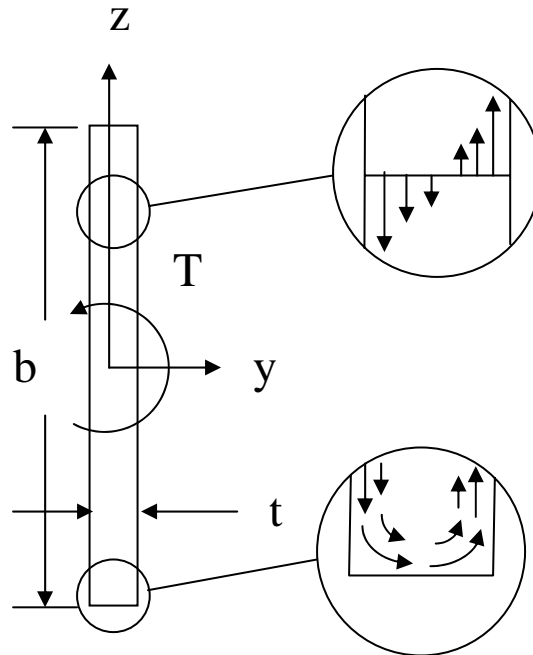


Fig. 7 Stress distributions for the torsion of a thin rectangle

On the two long faces of the section the warping function must satisfy the boundary conditions

$$\frac{\partial \psi}{\partial y} \Big|_{y=\pm t/2} = z \quad (26)$$

Since the other two faces of the cross section at $z = \pm b / 2$ are so small, we will assume we can treat the cross section as if were infinitely long and simply ignore the boundary conditions at those faces. In that case, we see that a warping function given by

$$\psi = yz \quad (27)$$

satisfies both Laplace's equation and the boundary conditions of Eq. (26), so it appears to solve this problem. The stresses generated by this solution are given by Eq.(25b) as

$$\begin{aligned} \sigma_{xz} &= 2G\phi'y \\ \sigma_{xy} &= 0 \end{aligned} \quad (28)$$

which produces the linear distribution of stress across the thickness shown in Fig. 7. This distribution does represent well the actual stress distribution over most of the cross section, although near the ends its is obviously in error since the shear stress must change direction and "flow" around the ends so as to remain tangent to the boundary at the edge of the cross section. If we compute the effective area moment from Eq. (25f) we find

$$J_{eff} = 4 \int_A y^2 dA = 4I_z = \frac{bt^3}{3} \quad (29)$$

since the maximum shear stress occurs at $y = \pm t / 2$, from Eq. (25f) also we find

$$c_{eff} = \pm 2y \Big|_{y=\pm t/2} = t$$

so that the maximum shear stress is given by

$$\tau_{max} = \frac{Tt}{J_{eff}} \quad (30)$$

We should note that if we had used the expression given in Eq. (20) instead to calculate the effective area moment, or if we had used the shear stresses of Eq. (28) in the equation

$$T = \int_A (y\sigma_{xz} - z\sigma_{xy}) dA \quad (31)$$

to calculate the effective area moment, we would come up with values smaller by a factor of two than the one given by Eq. (29) – the correct value ! This discrepancy occurs because of the fact that we ignored the behavior of the stresses at the ends and those end stresses are important when using either Eq.(20) or Eq. (31)., but can indeed be neglected when using Eq. (25f). Thus, when using approximate expression in calculating such torsional constants, all forms are not equally good for such approximations (although they are equivalent for exact solutions). We will discuss this point again later.

Although the torsion of other non-circular bars can be analyzed by solving the boundary value problem and associated quantities in Eqs. (25-a-f) the form of the boundary conditions that must be satisfied makes this approach rather difficult for other cases. Instead, a Prandtl stress function approach, as discussed in the next section, is more commonly used.