Stress-strain relations

General anisotropic material

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

[C] matrix is symmetric so there are 21 independent material constants

Orthotropic material

Wood, composites

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

Nine independent constants

Cubic material

Aluminum

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

3 independent constants

Isotropic material

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\left(C_{11}-C_{12}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\left(C_{11}-C_{12}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\left(C_{11}-C_{12}\right) \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

2 independent constants

$$C_{11} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}, \quad C_{12} = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

$$\frac{1}{2}\left(C_{11} - C_{12}\right) = \frac{E}{2(1+\nu)} = G$$

Generalized Hooke's law for an isotropic material

$$\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)e_{xx} + \nu\left(e_{yy} + e_{zz}\right)\right]$$

$$\sigma_{yy} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)e_{yy} + \nu\left(e_{xx} + e_{zz}\right)\right]$$

$$\sigma_{zz} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)e_{zz} + \nu\left(e_{xx} + e_{yy}\right)\right]$$

$$\sigma_{xy} = G\gamma_{xy}$$

$$\sigma_{xz} = G\gamma_{xz}$$

$$\sigma_{yz} = G\gamma_{yz}$$

$$G = \frac{E}{2(1+\nu)}$$

Generalized Hooke's law for an isotropic material

$$e_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right]$$

$$e_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right]$$

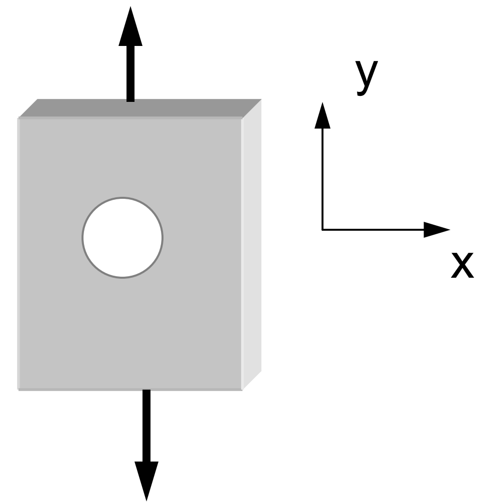
$$e_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right]$$

$$\gamma_{xy} = \frac{\sigma_{xy}}{G}$$

$$\gamma_{xz} = \frac{\sigma_{xz}}{G}$$

$$\gamma_{yz} = \frac{\sigma_{yz}}{G}$$

Plane Stress



$$\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$$

$$\sigma_{xx} = \sigma_{xx}(x, y)$$

$$\sigma_{yy} = \sigma_{yy}(x, y)$$

$$\sigma_{xy} = \sigma_{xy}(x, y)$$

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left[e_{xx} + \nu e_{yy}\right]$$

$$\sigma_{yy} = \frac{E}{1-\nu^2}\left[e_{yy} + \nu e_{xx}\right]$$

$$\sigma_{xy} = G\gamma_{xy}$$

$$e_{zz} = \frac{1}{E}\left[\cancel{\sigma_{zz}} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right]$$

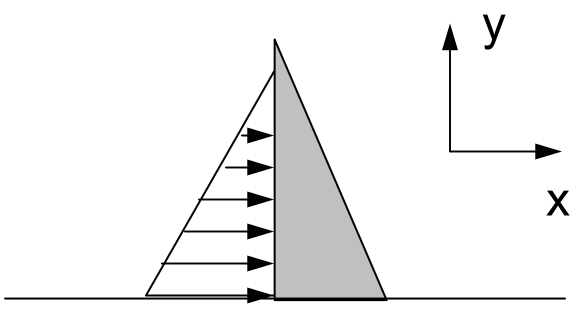
$$= \frac{-\nu}{(1-\nu)}\left[e_{xx} + e_{yy}\right]$$

Plane Strain

$$u_z = 0$$
$$u_x = u_x(x, y)$$
$$u_y = u_y(, y)$$

$\Rightarrow$

$$e_{zz} = \gamma_{xz} = \gamma_{yz} = 0$$
$$e_{xx} = e_{xx}(x, y)$$
$$e_{yy} = e_{yy}(x, y)$$
$$\gamma_{xy} = \gamma_{xy}(x, y)$$



$$\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)e_{xx} + \nu e_{yy}\right]$$
$$\sigma_{yy} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)e_{yy} + \nu e_{xx}\right]$$
$$\sigma_{xy} = G\gamma_{xy}$$

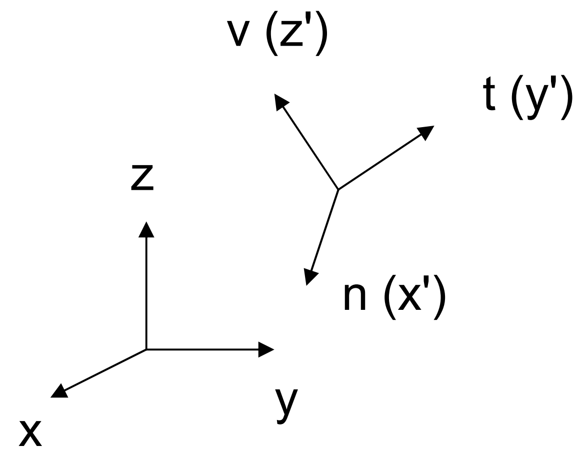
$$\sigma_{zz} = \frac{E\nu}{(1+\nu)(1-2\nu)}\left[e_{xx} + e_{yy}\right]$$

Elastic Constants (Stiffness matrix)

$$\{\sigma\} = [C]\{e\}$$

$$\{\sigma'\} = [C']\{e'\}$$

$$[C'] = [M][C][M]^T$$



$$[M] = \begin{bmatrix} l_{11}^2 & l_{21}^2 & l_{31}^2 & 2l_{21}l_{31} & 2l_{31}l_{11} & 2l_{21}l_{11} \\ l_{12}^2 & l_{22}^2 & l_{32}^2 & 2l_{22}l_{32} & 2l_{12}l_{32} & 2l_{12}l_{22} \\ l_{13}^2 & l_{23}^2 & l_{33}^2 & 2l_{23}l_{33} & 2l_{13}l_{33} & 2l_{13}l_{23} \\ l_{12}l_{13} & l_{22}l_{23} & l_{32}l_{33} & l_{22}l_{33}+l_{32}l_{23} & l_{12}l_{33}+l_{32}l_{13} & l_{12}l_{23}+l_{22}l_{13} \\ l_{11}l_{13} & l_{21}l_{23} & l_{31}l_{33} & l_{21}l_{33}+l_{31}l_{23} & l_{11}l_{33}+l_{31}l_{13} & l_{11}l_{23}+l_{21}l_{13} \\ l_{11}l_{12} & l_{21}l_{22} & l_{31}l_{32} & l_{21}l_{32}+l_{31}l_{22} & l_{11}l_{32}+l_{31}l_{12} & l_{11}l_{22}+l_{21}l_{12} \end{bmatrix}$$