Generalized Hooke's Law for Anisotropic materials

In the most general case, if we assume that the stresses are linearly related to the strains, we can write this relationship in matrix form as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

where, in the 6x6 stiffness C matrix there are a total of 21 elastic constants since the matrix is symmetric, i.e. $C_{12} = C_{21}, C_{23} = C_{32}$, etc. Most anisotropic materials used in engineering practice, however, have certain directional symmetries that considerably reduce the number of independent constants. Below, we list some of the important special cases.

Orthotropic material (9 constants)

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

This can also be turned around to write the strains in terms of the stresses as

$$\begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ D_{12} & D_{22} & D_{23} & 0 & 0 & 0 \\ D_{13} & D_{23} & D_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix}$$

which is usually written in the form

$$e_{xx} = \frac{1}{E_x}\sigma_{xx} - \frac{\nu_{yx}}{E_y}\sigma_{yy} - \frac{\nu_{zx}}{E_z}\sigma_{zz}$$
$$e_{yy} = \frac{1}{E_y}\sigma_{yy} - \frac{\nu_{xy}}{E_x}\sigma_{xx} - \frac{\nu_{zy}}{E_z}\sigma_{zz}$$
$$e_{zz} = \frac{1}{E_z}\sigma_{zz} - \frac{\nu_{xz}}{E_x}\sigma_{xx} - \frac{\nu_{yz}}{E_y}\sigma_{yy}$$
$$\gamma_{yz} = \frac{1}{G_{yz}}\sigma_{yx}$$
$$\gamma_{xz} = \frac{1}{G_{xz}}\sigma_{xz}$$
$$\gamma_{xy} = \frac{1}{G_{xy}}\sigma_{xy}$$

where

$$\frac{\nu_{xy}}{E_x} = \frac{\nu_{yx}}{E_y}, \frac{\nu_{xz}}{E_x} = \frac{\nu_{zx}}{E_z}, \frac{\nu_{yz}}{E_y} = \frac{\nu_{zy}}{E_z}$$

For plane stress $(\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0)$ the stress-strain relations for an orthotropic solid reduce to the form

$$\sigma_{xx} = \frac{E_x}{1-\nu_{xy}\nu_{yx}}\left(e_{xx} + \nu_{yx}e_{yy}\right)$$
$$\sigma_{yy} = \frac{E_x}{1-\nu_{xy}\nu_{yx}}\left(e_{yy} + \nu_{xy}e_{xx}\right)$$
$$\sigma_{xy} = G_{xy}\gamma_{xy}$$

Cubic material (3 constants)

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

Isotropic material (2 constants)

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{yz}\\ \sigma_{xz}\\ \sigma_{xy}\end{Bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{12} & 0 & 0 & 0\\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0\\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{2}\left(C_{11}-C_{12}\right) & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{2}\left(C_{11}-C_{12}\right) & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\left(C_{11}-C_{12}\right)\end{bmatrix}\begin{Bmatrix}e_{xx}\\ e_{yy}\\ e_{zz}\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{Bmatrix}$$

where

$$C_{11}=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)},\ C_{12}=\frac{E\nu}{(1+\nu)(1-2\nu)}$$

$$\frac{1}{2}\left(C_{11}-C_{12}\right)=\frac{E}{2(1+\nu)}=G$$

If these stress-strain relations are written out explicitly, we have

$$\sigma_{xx}=\frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)e_{xx}+\nu\left(e_{yy}+e_{zz}\right)\right]$$

$$\sigma_{yy}=\frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)e_{yy}+\nu\left(e_{xx}+e_{zz}\right)\right]$$

$$\sigma_{zz}=\frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)e_{zz}+\nu\left(e_{xx}+e_{yy}\right)\right]$$

$$\sigma_{xy}=G\gamma_{xy}$$

$$\sigma_{xz}=G\gamma_{xz}$$

$$\sigma_{yz}=G\gamma_{yz}$$

or, if we write instead the strains in terms of the stresses

$$
\begin{aligned}
e_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right] \\
e_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right] \\
e_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] \\
\gamma_{xy} &= \frac{\sigma_{xy}}{G} \\
\gamma_{xz} &= \frac{\sigma_{xz}}{G} \\
\gamma_{yz} &= \frac{\sigma_{yz}}{G}
\end{aligned}
$$

In the case of plane stress, $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$. Thus, setting the equation for σ_{zz} equal to zero shows that

$$
e_{zz} = \frac{-\nu}{(1-\nu)}\left[e_{xx} + e_{yy}\right]
$$

so that placing thus expression into the other two normal stress relations allows us to write all the stresses in terms of $e_{xx}, e_{yy}, \gamma_{xy}$ only. Thus for plane stress

$$
\begin{aligned}
\sigma_{xx} &= \frac{E}{1-\nu^2}\left[e_{xx} + \nu e_{yy}\right] \\
\sigma_{yy} &= \frac{E}{1-\nu^2}\left[e_{yy} + \nu e_{xx}\right] \\
\sigma_{xy} &= G\gamma_{xy}
\end{aligned}
$$

In the case of plane strain, we can set $e_{zz} = \gamma_{xz} = \gamma_{yz} = 0$. In this case we have

$$
\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)e_{xx} + \nu e_{yy}\right] \\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)e_{yy} + \nu e_{xx}\right] \\
\sigma_{xy} &= G\gamma_{xy}
\end{aligned}
$$

but note that there is still a normal stress, σ_{zz} , given by

$$\sigma_{zz} = \frac{E\nu}{(1+\nu)(1-2\nu)}\left[e_{xx} + e_{yy}\right]$$

Transformation of the elastic constants C matrix

One important difference between the isotropic case and the anisotropic cases listed above is that while the matrix of coefficients for the isotropic case is the same for <u>any</u> orientation of the x,y, z axis this is not true in general for the other materials and the expressions given above are only valid for a particular set of material axes. We can obtain the transformation equations for these coefficients, which like the stress and strain transformations involve the direction cosines relating a pair of axes, as will be shown below.

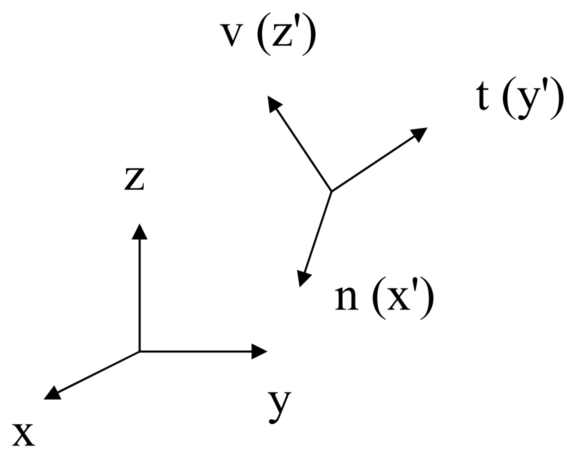
Another important difference between isotopic and anisotropic media is that the principal stress and principal strain directions <u>do not</u> coincide in general for anisotropic materials so that we need to calculate those directions (and the corresponding principal stress and strain values) separately for the stress and strain.

To obtain the transformation relations for the elastic constants, recall that we found that the stresses and (tensor) starins both transformed from one coordinate system to another according to the same rules, namely

$$\begin{aligned}[\sigma'] &= [l]^T[\sigma][l] \\ [e'] &= [l]^T[e][l]\end{aligned}$$

where $[l]$ is the 3x3 matrix of direction cosines (see figure below):

$$[l] = \begin{bmatrix} n_x & t_x & v_x \\ n_y & t_y & v_y \\ n_z & t_z & v_z \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$



If we write the stresses and strains in the original (unprimed) coordinates and in the rotated (primed) coordinates, we have

$$\{\sigma\} = [C]\{e\}$$
$$\{\sigma'\} = [C']\{e'\}$$

where here the stresses and strains are the column vectors shown previously. Using the stress and strain transformation relations and these stress-strain relations, after some considerable algebra one can show that

$$[C'] = [M][C][M]^T$$

where the 6x6 M matrix is given by

$$[M] = \begin{bmatrix} l_{11}^2 & l_{21}^2 & l_{31}^2 & 2l_{21}l_{31} & 2l_{31}l_{11} & 2l_{21}l_{11} \\ l_{12}^2 & l_{22}^2 & l_{32}^2 & 2l_{22}l_{32} & 2l_{12}l_{32} & 2l_{12}l_{22} \\ l_{13}^2 & l_{23}^2 & l_{33}^2 & 2l_{23}l_{33} & 2l_{13}l_{33} & 2l_{13}l_{23} \\ l_{12}l_{13} & l_{22}l_{23} & l_{32}l_{33} & l_{22}l_{33}+l_{32}l_{23} & l_{12}l_{33}+l_{32}l_{13} & l_{12}l_{23}+l_{22}l_{13} \\ l_{11}l_{13} & l_{21}l_{23} & l_{31}l_{33} & l_{21}l_{33}+l_{31}l_{23} & l_{11}l_{33}+l_{31}l_{13} & l_{11}l_{23}+l_{21}l_{13} \\ l_{11}l_{12} & l_{21}l_{22} & l_{31}l_{32} & l_{21}l_{32}+l_{31}l_{22} & l_{11}l_{32}+l_{31}l_{12} & l_{11}l_{22}+l_{21}l_{12} \end{bmatrix}$$