Strains in Cylindrical Coordinates

In class, we showed that the strain components relative to a set of (n, t, s) directions that were mutually orthogonal in the undeformed body could be expressed in terms of derivatives of the displacement vector, **u**, as:

$$\varepsilon_{nn} = \mathbf{n}\cdot\frac{\partial \mathbf{u}}{\partial s_n}$$

$$\varepsilon_{tt} = \mathbf{t}\cdot\frac{\partial \mathbf{u}}{\partial s_t}$$

$$\varepsilon_{ss} = \mathbf{s}\cdot\frac{\partial \mathbf{u}}{\partial s_s}$$

$$\gamma_{nt} = \mathbf{n}\cdot\frac{\partial \mathbf{u}}{\partial s_t} + \mathbf{t}\cdot\frac{\partial \mathbf{u}}{\partial s_n}\ , \quad etc.$$

Before, we chose n, t, and s along a set of Cartesian axes to obtain the strain components in rectangular coordinates. Similarly, we can use these expressions to find the strain components in any other coordinate system. Consider a set of cylindrical coordinate, for example:

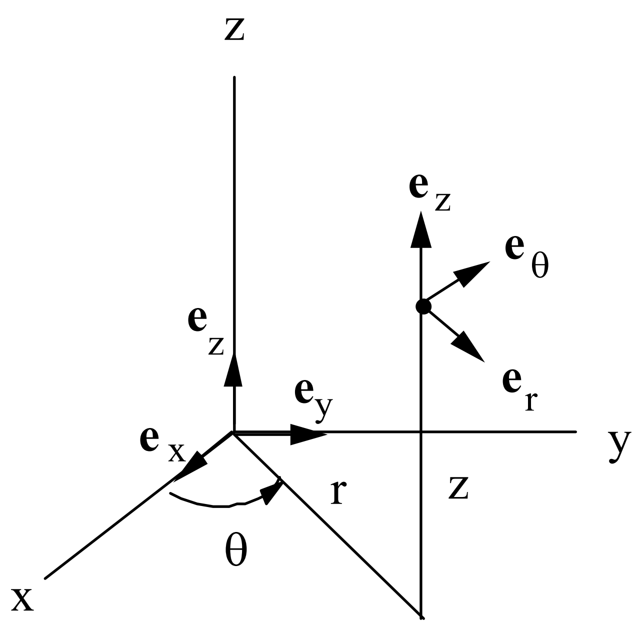

Figure 1. Cylindrical coordinates

In this case we can take

$$\partial s_n = \partial r$$
$$\partial s_t = r\partial\theta$$
$$\partial s_z = \partial z$$

and

$$\mathbf{n} = \mathbf{e}_r = \cos\theta\,\mathbf{e}_x + \sin\theta\,\mathbf{e}_y$$
$$\mathbf{t} = \mathbf{e}_\theta = -\sin\theta\,\mathbf{e}_x + \cos\theta\,\mathbf{e}_y$$
$$\mathbf{s} = \mathbf{e}_z$$

and where the displacement, **u**, is given by

$$\mathbf{u} = u_r\,\mathbf{e}_r + u_\theta\,\mathbf{e}_\theta + u_z\,\mathbf{e}_z$$

Consider, now the normal strain in the *r* direction. We have

$$\begin{aligned}\varepsilon_{rr} &= \mathbf{e}_r \cdot \frac{\partial \mathbf{u}}{\partial r} \\ &= \mathbf{e}_r \cdot \left(\frac{\partial u_r}{\partial r}\mathbf{e}_r + \frac{\partial u_\theta}{\partial r}\mathbf{e}_\theta + \frac{\partial u_z}{\partial r}\mathbf{e}_z \right) \\ &= \frac{\partial u_r}{\partial r}\end{aligned}$$

In the θ direction, however,

$$\begin{aligned}\varepsilon_{\theta\theta} &= \mathbf{e}_\theta \cdot \frac{1}{r}\frac{\partial \mathbf{u}}{\partial \theta} = \mathbf{e}_\theta \cdot \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta}\mathbf{e}_r + \frac{\partial u_\theta}{\partial \theta}\mathbf{e}_\theta + \frac{\partial u_z}{\partial \theta}\mathbf{e}_z + u_r\frac{\partial \mathbf{e}_r}{\partial \theta} + u_\theta\frac{\partial \mathbf{e}_\theta}{\partial \theta} \right) \\ &= \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\end{aligned}$$

where we have used the fact that

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta\,, \quad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r$$

The additional strain term coming from the radial displacement can be understood if we consider what such a displacement does in terms of the elongation of an element initially in the θ direction:

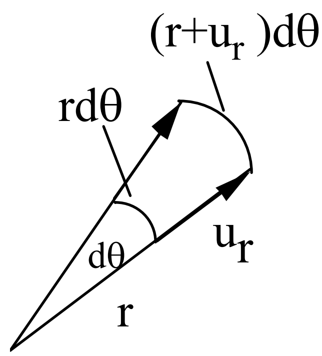

Figure 2. Effects of a radial displacement on an element in the θ direction

From Fig. 2 we see that a radial displacement causes an strain in the θ direction of an element initially in that direction given by

$$\left(\varepsilon_{\theta\theta}\right)_{u_r} = \frac{(r+u_r)\partial\theta - r\partial\theta}{r\partial\theta}$$
$$= \frac{u_r}{r}$$

Also, in the z direction we find

$$\varepsilon_{zz} = \mathbf{e}_z \cdot \frac{\partial \mathbf{u}}{\partial z} = \mathbf{e}_z \cdot \left(\frac{\partial u_r}{\partial z}\mathbf{e}_r + \frac{\partial u_\theta}{\partial z}\mathbf{e}_\theta + \frac{\partial u_z}{\partial z}\mathbf{e}_z \right)$$
$$= \frac{\partial u_z}{\partial z}$$

Finally, consider one if the shear strains, namely

$$\gamma_{r\theta} = \mathbf{e}_r \cdot \frac{1}{r}\frac{\partial \mathbf{u}}{\partial \theta} + \mathbf{e}_\theta \cdot \frac{\partial \mathbf{u}}{\partial r}$$
$$= \mathbf{e}_r \cdot \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta}\mathbf{e}_r + \frac{\partial u_\theta}{\partial \theta}\mathbf{e}_\theta + \frac{\partial u_z}{\partial \theta}\mathbf{e}_z + u_r\frac{\partial \mathbf{e}_r}{\partial \theta} + u_\theta\frac{\partial \mathbf{e}_\theta}{\partial \theta} \right)$$
$$+\mathbf{e}_\theta \cdot \left(\frac{\partial u_r}{\partial r}\mathbf{e}_r + \frac{\partial u_\theta}{\partial r}\mathbf{e}_\theta + \frac{\partial u_z}{\partial r}\mathbf{e}_z \right)$$
$$= \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}$$

The first two terms in this shear strain expression are analogous to the terms that appear in Cartesian coordinates. The last term can be understood by the fact that the displacement in the θ direction can itself cause a shear strain as shown in Figure 3:

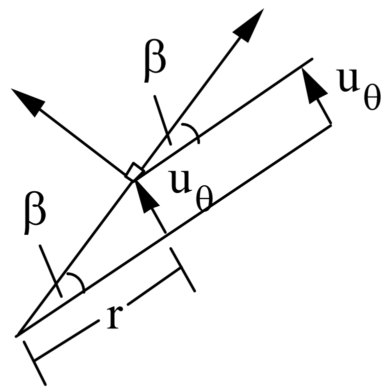

Figure 3. Shear strain caused by a displacement in the θ direction

where

$$(\gamma_{r\theta})_{u_\theta} = -\beta = -\frac{u_\theta}{r}$$

(the minus sign exists because a constant θ displacement causes the angle between two lines initially along the *r* and θ directions to be greater than ninety degrees as shown in Fig. 3.)