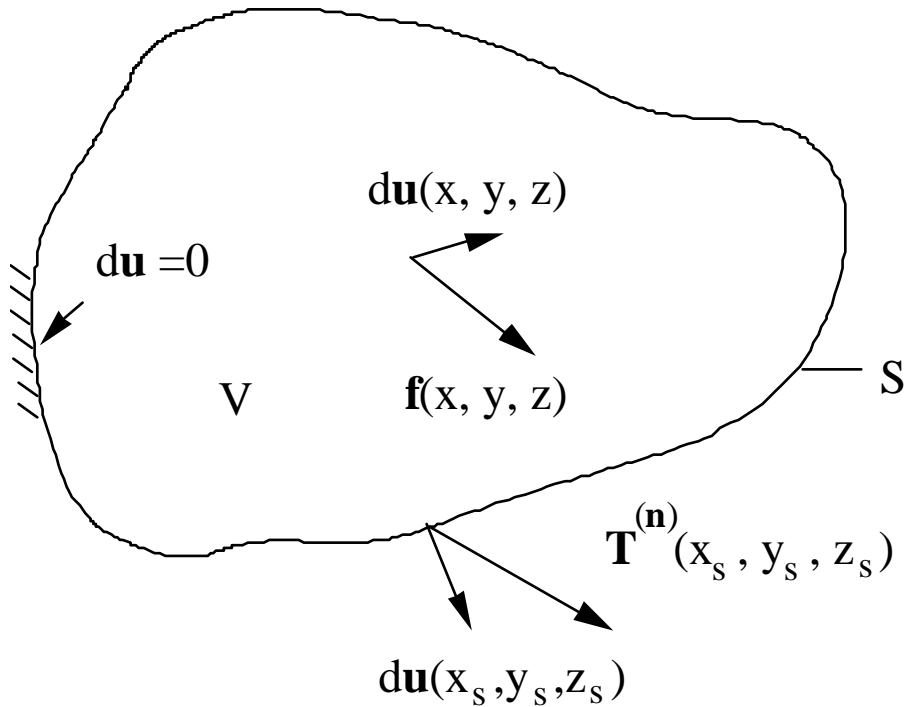


### Work, Strain Energy, and Stress-Strain Relations

Consider a body occupying a volume  $V$  whose surface is  $S$ . If this body experiences a small displacement  $d\mathbf{u}(x_1, x_2, x_3)$  the work done by the surface tractions and body force are given by

$$dW = \int_S \mathbf{T}^{(n)} \cdot d\mathbf{u} dS + \int_V \mathbf{f} \cdot d\mathbf{u} dV \quad (1)$$



If we express the traction vector in terms of the stress components and break both the displacement vector and the body force vector into their components, i.e.

$$\mathbf{T}^{(n)} = \sum_{i=1}^3 \sum_{j=1}^3 \sigma_{ij} n_i \mathbf{e}_j$$

$$\mathbf{f} = \sum_{j=1}^3 f_j \mathbf{e}_j$$

$$d\mathbf{u} = \sum_{j=1}^3 du_j \mathbf{e}_j$$

then

$$\mathbf{T}^{(n)} \cdot d\mathbf{u} = \sum_{i=1}^3 \sum_{j=1}^3 \sigma_{ij} n_i du_j$$

$$\mathbf{f} \cdot d\mathbf{u} = \sum_{j=1}^3 f_j du_j$$

which, when placed into Eq.(1), gives

$$dW = \sum_{i=1}^3 \sum_{j=1}^3 \int (\sigma_{ij} du_j) n_i dS + \sum_{j=1}^3 \int f_j du_j dV \quad (2)$$

However, the surface integral can be transformed into an integral over the volume V, since by Gauss' theorem (for any well behaved function g)

$$\int_S g n_i dS = \int_V \frac{\partial g}{\partial x_i} dV$$

so that we can rewrite Eq.(2) as

$$\begin{aligned} dW &= \sum_{j=1}^3 \int_V \left( \sum_{i=1}^3 \frac{\partial}{\partial x_i} (\sigma_{ij} du_j) + f_j du_j \right) dV \\ &= \sum_{j=1}^3 \int_V \left( \sum_{i=1}^3 \frac{\partial \sigma_{ij}}{\partial x_i} + f_j \right) du_j dV + \sum_{i=1}^3 \sum_{j=1}^3 \int \sigma_{ij} \frac{\partial}{\partial x_i} (du_j) dV \end{aligned} \quad (3)$$

The first volume integral in Eq. (3) vanishes because the equilibrium equations must be satisfied, and for the integrand of the second integral we have

$$\begin{aligned} \sum_{i=1}^3 \sum_{j=1}^3 \sigma_{ij} \frac{\partial}{\partial x_i} (du_j) &= \sum_{i=1}^3 \sum_{j=1}^3 \sigma_{ij} d \left( \frac{\partial u_j}{\partial x_i} \right) \\ &= \sum_{i=1}^3 \sum_{j=1}^3 \sigma_{ij} (de_{ij} + d\omega_{ij}) \\ &= \sum_{i=1}^3 \sum_{j=1}^3 \sigma_{ij} de_{ij} \end{aligned}$$

where the sum of terms involving the change of local rotation,  $d\omega_{ij}$  vanish because the stresses are symmetric while the rotation is antisymmetric, so that Eq. (3) becomes

$$dW = \int_V \left( \sum_{i=1}^3 \sum_{j=1}^3 \sigma_{ij} de_{ij} \right) dV \quad (4)$$

### Strain energy density

The work done by the external forces must be equal to the change of internal (strain) energy,  $U$ , of the body, i.e.

$$dW = dU = d \int_V u_0 dV = \int_V (du_0) dV \quad (5)$$

where  $u_0$  is the strain energy per unit volume. Comparing Eqs. (4) and (5) we see that the change in this *strain energy density* for the body must therefore be given by

$$du_0 = \sum_{i=1}^3 \sum_{j=1}^3 \sigma_{ij} de_{ij} \quad (6)$$

If we write the stresses in terms of the strains, we can express the strain energy density as a function of the strains only, which we can write symbolically as  $u_0 = u_0(e)$ , from which it follows

$$du_0(e) = \sum_{i=1}^3 \sum_{j=1}^3 \frac{\partial u_0(e)}{\partial e_{ij}} de_{ij} \quad (7)$$

so that comparing Eqs. (6) and (7) we find

$$\sigma_{ij} = \frac{\partial u_0(e)}{\partial e_{ij}} \quad (8)$$

### Linear elastic material

Equation (8) shows that if we can obtain the strain energy density for a material, then we can use that density to obtain the stress strain relations. In general the stress strain relationship will be nonlinear. However, if we assume the strains in the material are small, it makes sense to expand the strain energy in a power series. Keeping only quadratic terms at most we can write

$$u_0(e) = A + \sum_{i=1}^3 \sum_{j=1}^3 B_{ij} e_{ij} + \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 C_{ijkl} e_{ij} e_{kl}$$

If when the strains are all zero the strain energy is zero and the stresses are zero, then  $A = B_{ij} = 0$  and the strain energy is expressible entirely in terms of the elastic material

constants  $C_{ijkl}$ . Using Eq. (8) it follows then that the stress strain relationship is a purely linear one given by

$$\sigma_{ij} = \sum_{k=1}^3 \sum_{l=1}^3 C_{ijkl} e_{kl} \quad (i, j = 1, 2, 3) \quad (9)$$

Equation (9) looks very formidable since there are a total of 81 possible  $C_{ijkl}$  constants. Because of the symmetry of the stresses and strains, however, it follows that

$$C_{ijkl} = C_{jikl} = C_{ijlk}$$

These symmetries reduce the number of independent constants to 36. Moreover, since

$$C_{ijkl} = \frac{\partial^2 u_0(e)}{\partial e_{ij} \partial e_{kl}} = \frac{\partial^2 u_0(e)}{\partial e_{kl} \partial e_{ij}} = C_{klij} \quad (10)$$

the number of independent constants is reduced even further to a total of 21 for the most general linear elastic material. Because many of the  $C_{ijkl}$  terms are redundant, it is convenient to write the general stress strain relations in a more direct way as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \quad (11)$$

where  $C_{IJ}$  ( $I, J = 1, 2, 3, 4, 5, 6$ ) is related to  $C_{ijkl}$  through

$$\begin{aligned} I &= i \quad \text{if} \quad i = j \\ &= i + j + 1 \quad \text{if} \quad i \neq j \\ J &= k \quad \text{if} \quad k = l \\ &= k + l + 1 \quad \text{if} \quad k \neq l \end{aligned} \quad (12)$$

To further simplify these general stress strain relations we can express the stresses and strains in a reduced 6-dimensional vector form as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}, \quad \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{Bmatrix} \quad (13)$$

so that Eq. (11) becomes

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{Bmatrix} \quad (14)$$

which can be compactly expressed as

$$\sigma_I = \sum_{J=1}^6 C_{IJ} e_J \quad (I = 1, 2, \dots, 6) \quad (15)$$

Note that for a linear elastic material the strain energy density can be written as

$$u_0 = \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 C_{ijkl} e_{ij} e_{kl} = \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 \sigma_{ij} e_{ij} \quad (16)$$

or, more compactly, in the reduced notation, as

$$u_0 = \frac{1}{2} \sum_{I=1}^6 \sum_{J=1}^6 C_{IJ} e_I e_J = \frac{1}{2} \sum_{I=1}^6 \sigma_I e_I \quad (17)$$

and, similar to Eqs. (8) and (10) we have, in this reduced notation

$$\begin{aligned} \sigma_I &= \frac{\partial u_0}{\partial e_I} \quad (I = 1, 2, \dots, 6) \\ C_{IJ} &= \frac{\partial^2 u_0}{\partial e_I \partial e_J} = \frac{\partial^2 u_0}{\partial e_J \partial e_I} = C_{JI} \end{aligned} \quad (18)$$

so that the matrix of material coefficients,  $C_{IJ}$ , is symmetric.

**Isotropic, linear elastic material**

If a linear elastic material is also isotropic then the material coefficients can be expressed in terms of only two elastic constants such as Young's modulus,  $E$ , and Poisson's ratio,  $\nu$ , where

$$\begin{aligned} C_{11} = C_{22} = C_{33} &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \\ C_{12} = C_{21} = C_{13} = C_{31} = C_{23} = C_{32} &= \frac{E\nu}{(1+\nu)(1-2\nu)} \\ C_{44} = C_{55} = C_{66} &= \frac{1}{2}(C_{11} - C_{12}) = \frac{E}{2(1+\nu)} \equiv G \\ \text{all others} &= 0 \end{aligned} \quad (19)$$

so that

$$\begin{aligned} u_0(e) &= \frac{1}{2} \frac{E}{(1+\nu)(1-2\nu)} \left[ (1-\nu)(e_{xx}^2 + e_{yy}^2 + e_{zz}^2) + 2\nu(e_{xx}e_{yy} + e_{xx}e_{zz} + e_{yy}e_{zz}) \right] \\ &\quad + \frac{1}{2} G(\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2) \end{aligned}$$

or, in terms of principal strains

$$u_0(e) = \frac{1}{2} \frac{E}{(1+\nu)(1-2\nu)} \left[ (1-\nu)(e_{p1}^2 + e_{p2}^2 + e_{p3}^2) + 2\nu(e_{p1}e_{p2} + e_{p1}e_{p3} + e_{p2}e_{p3}) \right] \quad (20)$$

which can also be written in terms of the strain invariants,  $J_1, J_2$  as

$$u_0(e) = \frac{1}{2} \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} J_1^2 - 2GJ_2 \quad (21)$$

where

$$\begin{aligned} J_1 &= e_{p1} + e_{p2} + e_{p3} \\ J_2 &= e_{p1}e_{p2} + e_{p1}e_{p3} + e_{p2}e_{p3} \end{aligned}$$

If we write the strains in terms of the stresses then the strain energy density in terms of stresses only becomes

$$u_0(\sigma) = \frac{1}{2E} \left[ \sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 - 2\nu(\sigma_{xx}\sigma_{yy} + \sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz}) \right] + \frac{1}{2G} (\sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2) \quad (22)$$

In terms of the principal stresses we have

$$u_0(\sigma) = \frac{1}{2E} \left[ \sigma_{p1}^2 + \sigma_{p2}^2 + \sigma_{p3}^2 - 2\nu(\sigma_{p1}\sigma_{p2} + \sigma_{p1}\sigma_{p3} + \sigma_{p2}\sigma_{p3}) \right] \quad (23)$$

or for the stress invariants,  $I_1, I_2$

$$u_0(\sigma) = \frac{1}{2E} [I_1^2 - 2(1+\nu)I_2] = \frac{1}{2E} I_1^2 - \frac{1}{2G} I_2 \quad (24)$$

### Distortional Strain Energy

Strain energy concepts are very useful when discussing failure concepts, since intuitively we expect that a material will fail if we place too much energy into it. For failure with respect to slip (yielding), it has been observed experimentally that such failure cannot be produced by a hydrostatic state of stress, i.e. where we apply a pure pressure to the material. Since the strain energy in general contains both hydrostatic and distortional energy components, we need to first remove the hydrostatic part of the strain energy before we can use that energy in a failure theory (for slip). To accomplish the removal of the hydrostatic component, note that for a stress state consisting of a pure pressure  $p$  we have  $\sigma_{p1} = \sigma_{p2} = \sigma_{p3} = -p$  and hence the hydrostatic part of the strain energy density is

$$u_0 = u_p = \frac{-3(1-2\nu)}{2E} p^2$$

For a general a state of stress, if we replace the pressure  $p$  by the average of the three principal stresses, i.e. set

$$p = -\frac{\sigma_{p1} + \sigma_{p2} + \sigma_{p3}}{3}$$

then the hydrostatic component is

$$u_p = \frac{(1-2\nu)}{6E} (\sigma_{p1} + \sigma_{p2} + \sigma_{p3})^2 \quad (25)$$

Using this result we now define the *distortional strain energy density*,  $u_d$ , as

$$u_d = u_0 - u_p \quad (26)$$

Using Eqs. (23) and (25), after some algebra we find

$$\begin{aligned} u_d &= \frac{(1+\nu)}{6E} \left[ (\sigma_{p1} - \sigma_{p2})^2 + (\sigma_{p1} - \sigma_{p3})^2 + (\sigma_{p2} - \sigma_{p3})^2 \right] \\ &= \frac{3(1+\nu)}{2E} |(\tau_s)_{oct}|^2 \end{aligned}$$

### Complimentary strain energy density

In applying work-energy principles, a quantity called the complimentary strain energy per unit volume,  $u_0^c$ , also appears, where

$$u_0^c = \sum_{i=1}^3 \sum_{j=1}^3 \sigma_{ij} e_{ij} - u_0 \quad (27)$$

A small (differential) change of this *complimentary strain energy density* is therefore given by

$$du_0^c = \sum_{i=1}^3 \sum_{j=1}^3 e_{ij} d\sigma_{ij} \quad (28)$$

If we now express the strains as functions of the stresses so that the complimentary strain energy can be considered as a function of the stresses only, we have

$$du_0^c(\sigma) = \sum_{i=1}^3 \sum_{j=1}^3 \frac{\partial u_0^c(\sigma)}{\partial \sigma_{ij}} d\sigma_{ij} \quad (29)$$

so that comparing Eqs.(28) and (29) we find

$$e_{ij} = \frac{\partial u_0^c(\sigma)}{\partial \sigma_{ij}} \quad (30)$$

Compare Eqs. (8) and (30). It follows from those two equations that  $u_0$  and  $u_0^c$  play similar (complimentary) roles where the stresses and strains are concerned - hence the name complimentary strain energy density for  $u_0^c$ .

### A linear elastic material

If we have a linear elastic material, it follows from Eqs. (16) and (27) that  $u_0^c = u_0$ , and hence

$$u_0(\sigma, e) = u_0^c(\sigma, e) = \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 \sigma_{ij} e_{ij} \quad (31)$$

Similarly, all the other expressions obtained for the strain energy density also are valid for the complimentary strain energy as well.

### Strain Energy for Axial Loads, Bending, and Torsion

The previous sections have discussed the concept of strain energy in general. This section will obtain explicit expressions for the total strain energy for some the simple loading cases normally treated by a strength of materials approach.

#### Axial Loads

In this case we have

$$\sigma_{xx} = \frac{P(x)}{A}$$

$$e_{xx} = \frac{du_x}{dx}$$

$$\sigma_{xx} = E e_{xx}$$

so that in terms of stresses or loads the total strain energy is

$$\begin{aligned} U &= \frac{1}{2} \int_0^L \frac{\sigma_{xx}^2}{E} A dx \\ &= \frac{1}{2} \int_0^L \frac{[P(x)]^2}{AE} dx \end{aligned}$$

Similarly, in terms of strains or displacements

$$\begin{aligned} U &= \frac{1}{2} \int_0^L E e_{xx}^2 A dx \\ &= \frac{1}{2} \int_0^L EA \left( \frac{du_x}{dx} \right)^2 dx \end{aligned}$$

#### Bending

For the flexure stress and strain we have

$$\sigma_{xx} = \frac{-M(x)z}{I}$$

$$e_{xx} = \frac{-M(x)z}{EI} = -\frac{d^2w(x)}{dx^2}z$$

which gives the total strain energy as a function of the bending moment as

$$U = \int_0^L \int_A \frac{\sigma_{xx}^2}{2E} dA dx$$

$$= \frac{1}{2} \int_0^L \left\{ \int_A z^2 dA \right\} \frac{M(x)^2}{EI^2} dx$$

$$= \frac{1}{2} \int_0^L \frac{M(x)^2}{EI} dx$$

or, in terms of the deflection of the neutral axis

$$U = \frac{1}{2} \int_0^L \int_A E e_{xx}^2 dA dx$$

$$= \frac{1}{2} \int_0^L \left\{ \int_A z^2 dA \right\} E \left( \frac{d^2w(x)}{dx^2} \right)^2 dx$$

$$= \frac{1}{2} \int_0^L EI \left( \frac{d^2w(x)}{dx^2} \right)^2 dx$$

For the shear stress in bending we have the strength of materials expression

$$\tau = \frac{V(x)Q(z)}{I t(z)}$$

so that

$$U = \int_0^L \int_A \frac{\tau^2}{2G} dA dx$$

$$= \frac{1}{2} \int_0^L \left\{ \frac{A}{I^2} \int_A \frac{Q(z)^2}{t(z)^2} dA \right\} \frac{V(x)^2}{GA} dx$$

$$= \frac{1}{2} \int_0^L k \frac{V(x)^2}{GA} dx$$

where the dimensionless constant  $k$  is defined as

$$k = \frac{A}{I^2} \int_A \frac{Q(z)^2}{t(z)^2} dA$$

This constant is a function of the geometry of the cross section only. For example, for a rectangular cross section  $k = 1.2$ .

In many bending problems involving long and slender beams, the strain energy associated with the flexure stresses is considerably larger than that for the shear stresses. Thus, in these cases the strain energy of the shear stresses can be neglected.

### Torsion

Recall that the shear stress in torsion for a circular cross section is given by

$$\begin{aligned}\tau &= \frac{T(x)r}{J} \\ \gamma &= \frac{d\phi(x)}{dx} r \\ \tau &= G\gamma\end{aligned}$$

Using these results the strain energy in terms of the torque is

$$\begin{aligned}U &= \int_0^L \int_A \frac{\tau^2}{2G} dA dx \\ &= \frac{1}{2} \int_0^L \left\{ \int_A r^2 dA \right\} \frac{T(x)^2}{GJ^2} dx \\ &= \frac{1}{2} \int_0^L \frac{T(x)^2}{GJ} dx\end{aligned}$$

or, in terms of the twist per unit length

$$\begin{aligned}U &= \int_0^L \int_A \frac{G\gamma^2}{2} dA dx \\ &= \frac{1}{2} \int_0^L \left\{ \int_A r^2 dA \right\} G \left( \frac{d\phi(x)}{dx} \right)^2 dx \\ &= \frac{1}{2} \int_0^L GJ \left( \frac{d\phi(x)}{dx} \right)^2 dx\end{aligned}$$