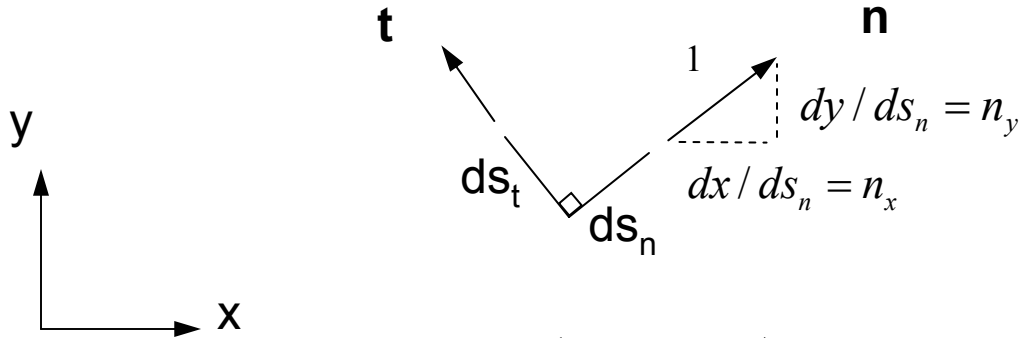


# Strain Transformations (plane strain)



$$e_{nt} = \frac{1}{2} \left( \frac{\partial u_n}{\partial s_t} + \frac{\partial u_t}{\partial s_n} \right)$$

$$= \frac{1}{2} \left( \mathbf{n} \cdot \frac{\partial \mathbf{u}}{\partial s_t} + \mathbf{t} \cdot \frac{\partial \mathbf{u}}{\partial s_n} \right)$$

$u_n$  ... displacement in  $\mathbf{n}$  direction  
 $u_t$  ... displacement in  $\mathbf{t}$  direction

$$e_{nt} = \frac{1}{2} \left( \mathbf{n} \cdot \frac{\partial \mathbf{u}}{\partial s_t} + \mathbf{t} \cdot \frac{\partial \mathbf{u}}{\partial s_n} \right)$$

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial s_n} &= \frac{\partial \mathbf{u}}{\partial x} \frac{dx}{ds_n} + \frac{\partial \mathbf{u}}{\partial y} \frac{dy}{ds_n} \\ &= \frac{\partial \mathbf{u}}{\partial x} n_x + \frac{\partial \mathbf{u}}{\partial y} n_y \end{aligned}$$

In the x,y coordinates  $\mathbf{u} = u_x \mathbf{e}_x + u_y \mathbf{e}_y$  so we have

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial x} n_x &= \frac{\partial u_x}{\partial x} n_x \mathbf{e}_x + \frac{\partial u_y}{\partial x} n_x \mathbf{e}_y \\ \frac{\partial \mathbf{u}}{\partial y} n_y &= \frac{\partial u_x}{\partial y} n_y \mathbf{e}_x + \frac{\partial u_y}{\partial y} n_y \mathbf{e}_y \end{aligned}$$

and hence  $\frac{\partial \mathbf{u}}{\partial s_n} = \left( \frac{\partial u_x}{\partial x} n_x + \frac{\partial u_x}{\partial y} n_y \right) \mathbf{e}_x + \left( \frac{\partial u_y}{\partial x} n_x + \frac{\partial u_y}{\partial y} n_y \right) \mathbf{e}_y$

so that with  $\mathbf{t} = t_x \mathbf{e}_x + t_y \mathbf{e}_y$

we find  $\mathbf{t} \cdot \frac{\partial \mathbf{u}}{\partial s_n} = \left( \frac{\partial u_x}{\partial x} n_x + \frac{\partial u_x}{\partial y} n_y \right) t_x + \left( \frac{\partial u_y}{\partial x} n_x + \frac{\partial u_y}{\partial y} n_y \right) t_y$

$$e_{nt} = \frac{1}{2} \left( \mathbf{n} \cdot \frac{\partial \mathbf{u}}{\partial s_t} + \mathbf{t} \cdot \frac{\partial \mathbf{u}}{\partial s_n} \right)$$

$$\mathbf{t} \cdot \frac{\partial \mathbf{u}}{\partial s_n} = \left( \frac{\partial u_x}{\partial x} n_x + \frac{\partial u_x}{\partial y} n_y \right) t_x + \left( \frac{\partial u_y}{\partial x} n_x + \frac{\partial u_y}{\partial y} n_y \right) t_y$$

similarly we can show

$$\mathbf{n} \cdot \frac{\partial \mathbf{u}}{\partial s_t} = \left( \frac{\partial u_x}{\partial x} t_x + \frac{\partial u_x}{\partial y} t_y \right) n_x + \left( \frac{\partial u_y}{\partial x} t_x + \frac{\partial u_y}{\partial y} t_y \right) n_y$$

so that

$$e_{nt} = \frac{1}{2} \left[ 2 \frac{\partial u_x}{\partial x} n_x t_x + 2 \frac{\partial u_y}{\partial y} n_y t_y \right. \\ \left. + t_x n_y \left( \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \right) \right. \\ \left. + t_y n_x \left( \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \right) \right]$$

or, in terms of strains

$$e_{nt} = e_{xx} n_x t_x + e_{yy} n_y t_y + e_{xy} (n_x t_y + n_y t_x)$$

$$e_{nt} = e_{xx} n_x t_x + e_{yy} n_y t_y + e_{xy} (n_x t_y + n_y t_x)$$

For the general 3-D strain case

$$\begin{aligned} e_{nt} = & e_{xx} n_x t_x + e_{yy} n_y t_y + e_{zz} n_z t_z \\ & + e_{xy} (n_x t_y + n_y t_x) \\ & + e_{xz} (n_x t_z + n_z t_x) \\ & + e_{yz} (n_y t_z + n_z t_y) \end{aligned}$$

## Principal Strains, Maximum shear

principal strains

$$e_p^3 - J_1 e_p^2 + J_2 e_p - J_3 = 0$$

$$J_1 = e_{xx} + e_{yy} + e_{zz}$$

$$J_2 = \begin{vmatrix} e_{xx} & e_{xy} \\ e_{xy} & e_{yy} \end{vmatrix} + \begin{vmatrix} e_{xx} & e_{xz} \\ e_{xz} & e_{zz} \end{vmatrix} + \begin{vmatrix} e_{zz} & e_{zy} \\ e_{zy} & e_{yy} \end{vmatrix}$$

$$J_3 = \begin{vmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{vmatrix}$$

principal strain directions

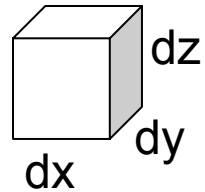
$$(e_{xx} - e_p)n_x + e_{xy}n_y + e_{xz}n_z = 0$$

$$e_{yx}n_x + (e_{yy} - e_p)n_y + e_{yz}n_z = 0$$

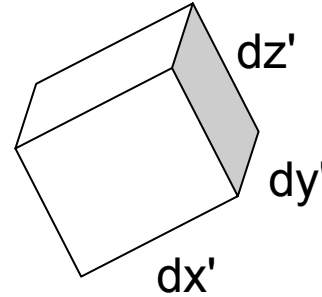
$$e_{zx}n_x + e_{zy}n_y + (e_{zz} - e_p)n_z = 0$$

$$|\gamma|_{\max} = \max \begin{cases} |e_{p1} - e_{p2}| \\ |e_{p1} - e_{p3}| \\ |e_{p2} - e_{p3}| \end{cases}$$

## Dilatational (volumetric) strain



undeformed body



deformed body

$$dx' = dx + e_{xx} dx$$

$$dy' = dy + e_{yy} dy$$

$$dz' = dz + e_{zz} dz$$

$$dx' dy' dz' = dx dy dz + (e_{xx} + e_{yy} + e_{zz}) dx dy dz + H.O.T.$$

so 
$$dV' - dV = (e_{xx} + e_{yy} + e_{zz}) dV$$

or 
$$e_{xx} + e_{yy} + e_{zz} = \frac{dV' - dV}{dV} \quad \text{volumetric strain}$$

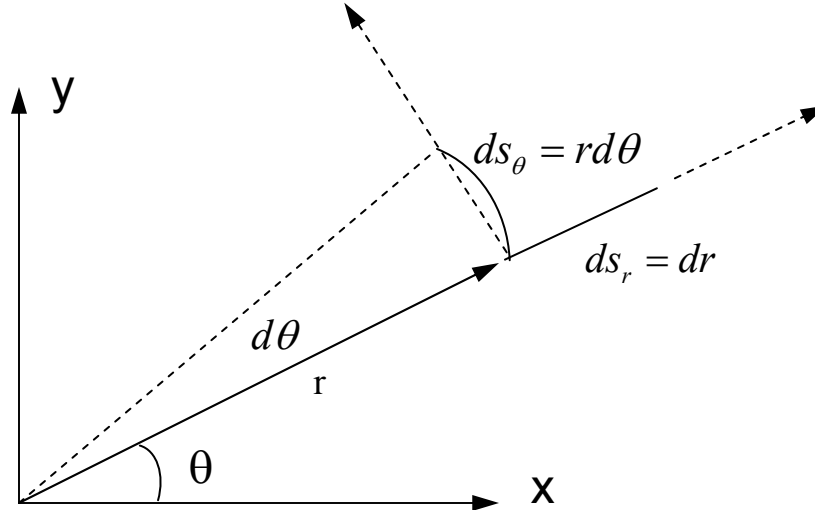
# Strain-Displacement Relations in Polar Coordinates (plane strain)

In Cartesian Coordinates

$$e_{xx} = \frac{\partial u_x}{\partial x}$$

$$e_{yy} = \frac{\partial u_y}{\partial y}$$

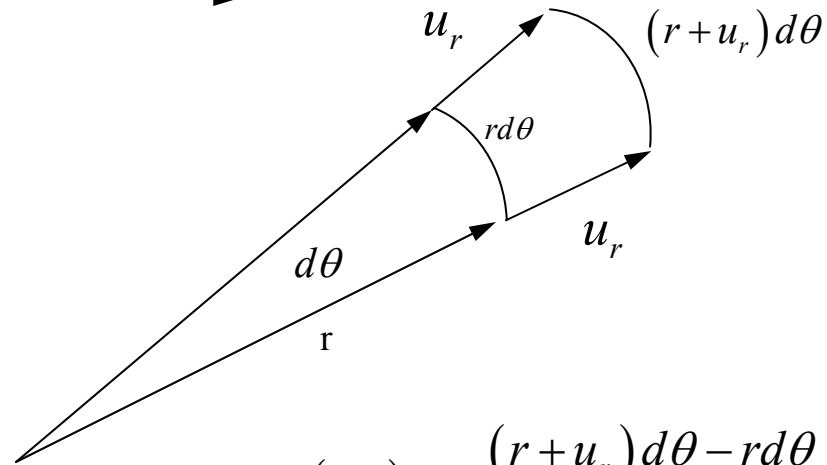
$$\gamma_{xy} = 2e_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$



$$e_{rr} = \frac{\partial u_r}{\partial s_r} = \frac{\partial u_r}{\partial r}$$

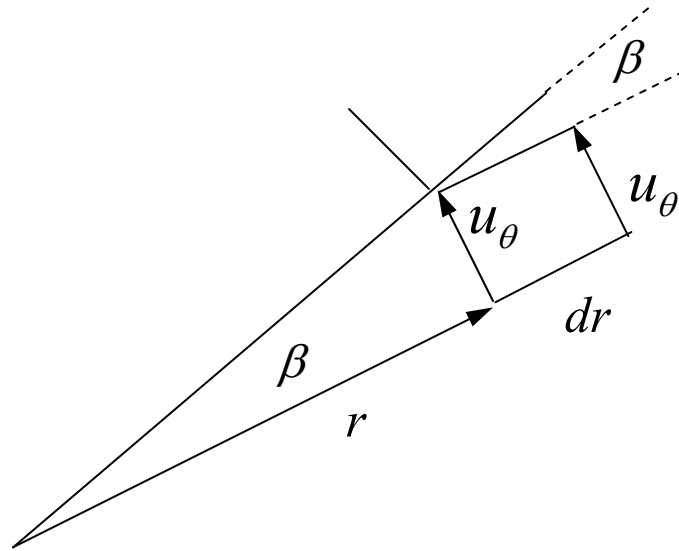
$$e_{\theta\theta} = \frac{\partial u_\theta}{\partial s_\theta} + (e_{\theta\theta})_{u_r}$$

$$= \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$$



$$(e_{\theta\theta})_{u_r} = \frac{(r + u_r) d\theta - r d\theta}{r d\theta} = \frac{u_r}{r}$$

$$\begin{aligned}\gamma_{r\theta} &= \frac{\partial u_r}{\partial s_\theta} + \frac{\partial u_\theta}{\partial s_r} + (\gamma_{r\theta})_{\text{constant } u_\theta} \\ &= \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\end{aligned}$$



$$(\gamma_{r\theta})_{\text{constant } u_\theta} = -\beta = -\frac{u_\theta}{r}$$

Summary

$$e_{rr} = \frac{\partial u_r}{\partial r}$$

$$e_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$$

$$\gamma_{r\theta} = \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}$$

## Strain Compatibility Equations (plane strain)

three strains

$$e_{xx}(x, y), e_{yy}(x, y), \gamma_{xy}(x, y)$$

two displacements

$$u_x(x, y), u_y(x, y)$$

The compatibility equation(s) guarantee that for a singly connected body (one with no holes) one can integrate the strains to obtain the displacements

$$e_{xx} = \frac{\partial u_x}{\partial x} \implies u_x = \int e_{xx} \partial x + f(y)$$

$$e_{yy} = \frac{\partial u_y}{\partial y} \implies u_y = \int e_{yy} \partial y + g(x)$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = \int \frac{\partial e_{xx}}{\partial y} \partial x + \int \frac{\partial e_{yy}}{\partial x} \partial y + \frac{df}{dy} + \frac{dg}{dx}$$

differentiating  
both sides on  
 $x, y$

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2}$$

compatibility equation

# Compatibility equations for 3-D state of strain

six strains

$$e_{xx}(x, y, z), e_{yy}(x, y, z), e_{zz}(x, y, z)$$

$$\gamma_{xy}(x, y, z), \gamma_{xz}(x, y, z), \gamma_{yz}(x, y, z)$$

three displacements

$$u_x(x, y, z), u_y(x, y, z), u_z(x, y, z)$$

$$\frac{\partial^2 \gamma_{yz}}{\partial y \partial z} - \frac{\partial^2 e_{yy}}{\partial z^2} - \frac{\partial^2 e_{zz}}{\partial y^2} = 0$$

$$\frac{\partial^2 \gamma_{xz}}{\partial x \partial z} - \frac{\partial^2 e_{zz}}{\partial x^2} - \frac{\partial^2 e_{xx}}{\partial z^2} = 0$$

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} - \frac{\partial^2 e_{xx}}{\partial y^2} - \frac{\partial^2 e_{yy}}{\partial x^2} = 0$$

$$2 \frac{\partial^2 e_{zz}}{\partial x \partial y} + \frac{\partial^2 \gamma_{xy}}{\partial z^2} - \frac{\partial^2 \gamma_{yz}}{\partial z \partial x} - \frac{\partial^2 \gamma_{xz}}{\partial z \partial y} = 0$$

$$2 \frac{\partial^2 e_{xx}}{\partial y \partial z} + \frac{\partial^2 \gamma_{yz}}{\partial x^2} - \frac{\partial^2 \gamma_{xz}}{\partial x \partial y} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial z} = 0$$

$$2 \frac{\partial^2 e_{yy}}{\partial z \partial x} + \frac{\partial^2 \gamma_{xz}}{\partial y^2} - \frac{\partial^2 \gamma_{xy}}{\partial y \partial z} - \frac{\partial^2 \gamma_{yz}}{\partial y \partial x} = 0$$

Six compatibility equations

(not all independent - only three independent relations)

single compatibility equation for plane strain or plane stress

