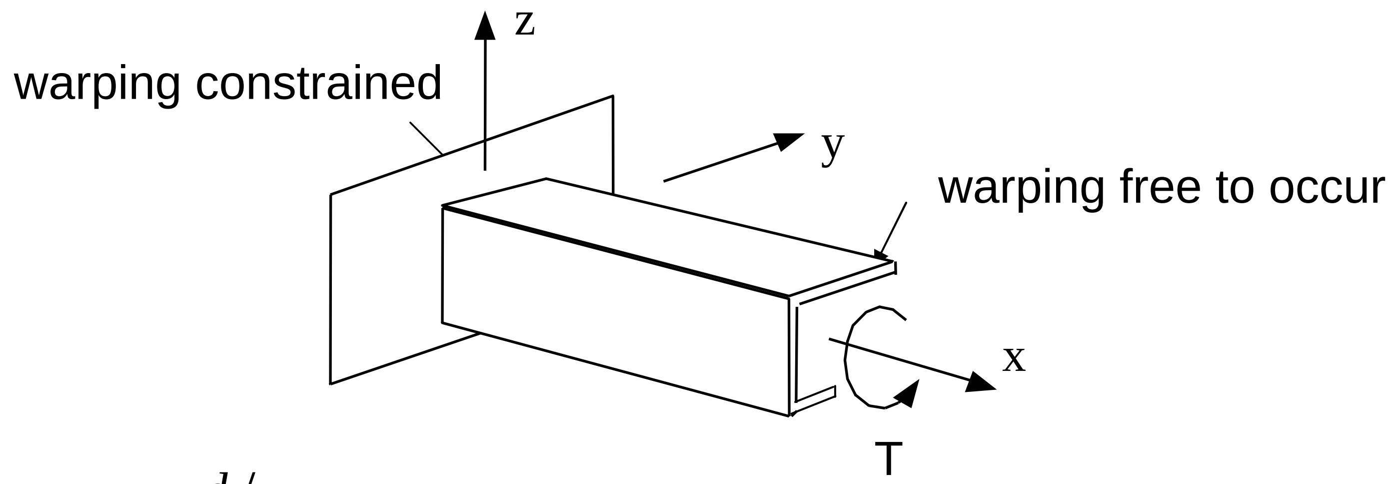
Restraint of Warping (open , thin sections)



$$u_x = -\frac{d\phi}{dx}\omega$$

if dϕ/dx is no longer a constant, this will cause a normal stress to develop

$$\sigma_{xx} = Ee_{xx} = E\frac{\partial u_x}{\partial x}$$

$$= -E\omega\frac{d^2\phi}{dx^2}$$

this normal stress must not produce any axial force or bending moments

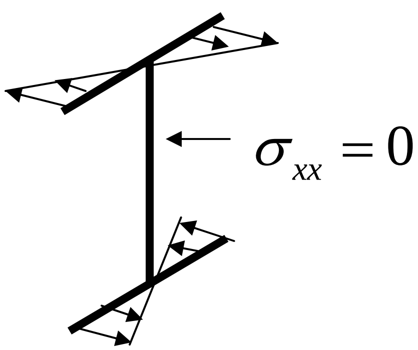
$$\int_A \sigma_{xx} dA = E\phi'' \int_A \omega \, dA = 0$$

$$\int_A y\sigma_{xx} dA = E\phi'' \int_A y\omega \, dA = 0$$

$$\int_A z\sigma_{xx} dA = E\phi'' \int_A z\omega \, dA = 0$$

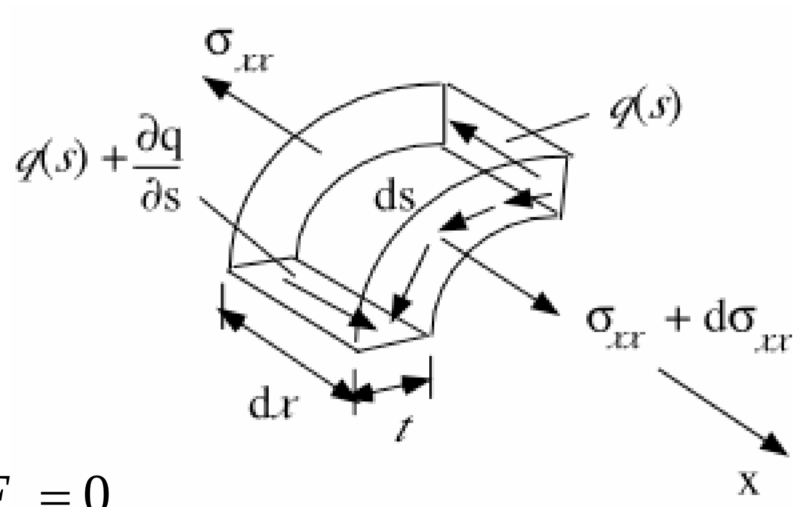
which says that the sectorial area function must be a <u>principal sectorial area function</u> and we have

$$\sigma_{xx} = -E\,\omega_p \frac{d^2\phi}{dx^2}$$



Note: these stresses are self-equilibrated but they do not rapidly decay from the ends (thus, they violate Saint Venant's principle)

However, a changing axial stress will require that additional shear flows be developed (recall how we obtained the VQ/It expression for shear stress in bending)



$$\sum F_x = 0$$

$$\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}dx\right)tds - \sigma_{xx}tds + \left(q(s) + \frac{\partial q}{\partial s}ds\right)dx - q(s)dx = 0$$

$$\frac{\partial q}{\partial s} = -t\frac{\partial \sigma_{xx}}{\partial x}$$

$$\Longrightarrow \quad = E\frac{d^3\phi}{dx^3}t\,\omega_p = E\frac{d^2\beta}{dx^2}t\,\omega_p \qquad \beta = \phi' = \frac{d\phi}{dx}$$

integrating gives

$$q(s) = E\frac{d^2\beta}{dx^2}\int_0^s \omega_p \, t ds$$

$$= E\frac{d^2\beta}{dx^2}\int \omega_p dA$$

which is the shear flow due to restraint of warping. The torque associated with this shear flow is

$$T_q = \int_{s=0}^{s=s_f} r_{\perp} q \, ds$$

$$= \int_0^{s_f} q \, d\omega_p$$

Thus, the total torque is

$$T = T_{SV} + T_q$$

$$= GJ_{eff}\beta + \int_0^{s_f} q \, d\omega_p$$

Integration by parts lets us express this in the form

$$T = GJ_{eff}\beta + q\omega_p\Big|_{s=0}^{s=s_f} - \int_0^{s_f}\omega_p\left(\frac{\partial q}{\partial s}\right)ds$$

$$= GJ_{eff}\beta - E\frac{d^2\beta}{dx^2}\int_A \omega_p^2\, dA$$

and defining $J_\omega = \int_A \omega_p^2\, dA$

gives

$$\frac{d^2\beta}{dx^2} - k^2\beta = -k^2\frac{T}{GJ_{eff}}$$

← differential equation for β(x)

where $k^2 = \dfrac{GJ_{eff}}{EJ_\omega}$

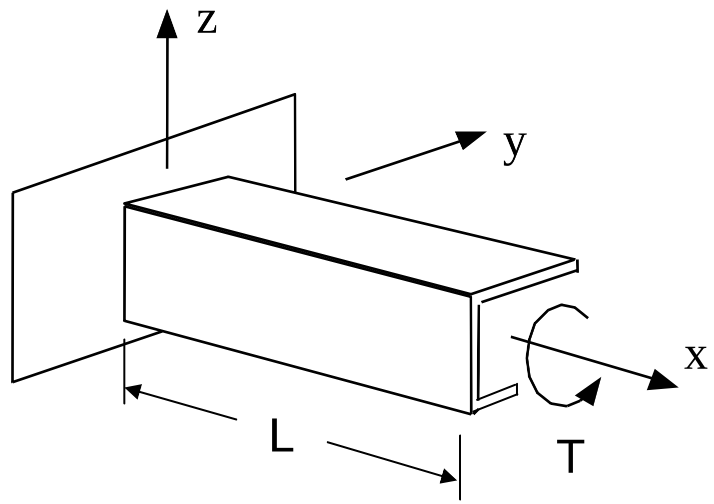
If the torque T = constant, solution is

$$\beta = C\sinh(kx) + D\cosh(kx) + \frac{T}{GJ_{eff}}$$

C, D are constants. We need to find these from the boundary conditions

Fixed end: $\beta = 0$ ($u_x = 0$)

Unconstrained end: $\frac{d\beta}{dx} = 0$ ($\sigma_{xx} = 0$)



$$\beta\Big|_{x=0} = 0 \quad \Rightarrow \quad D = -\frac{T}{GJ_{eff}}$$

$$\frac{d\beta}{dx}\bigg|_{x=L} = 0 \quad \Rightarrow \quad C = \frac{T}{GJ_{eff}}\tanh(kL)$$

which gives $$\beta = \frac{T}{GJ_{eff}}\left[\tanh(kL)\sinh(kx) - \cosh(kx) + 1\right]$$

$$\beta = \frac{d\phi}{dx} = \frac{T}{GJ_{eff}}\left[\tanh(kL)\sinh(kx) - \cosh(kx) + 1\right]$$

To get the twist we must integrate, using the additional boundary condition $\phi(0) = 0$. We obtain at x = L

$$\phi(L) = \int_0^L \beta \, dx = \frac{TL}{GJ_{eff}}\left[1 - \frac{\tanh(kL)}{kL}\right]$$

The axial stress in the bar is

$$\sigma_{xx} = -E\,\omega_p \frac{d\beta}{dx} = -E\,\omega_p \frac{Tk}{GJ_{eff}}\left[\tanh(kL)\cosh(kx) - \sinh(kx)\right]$$

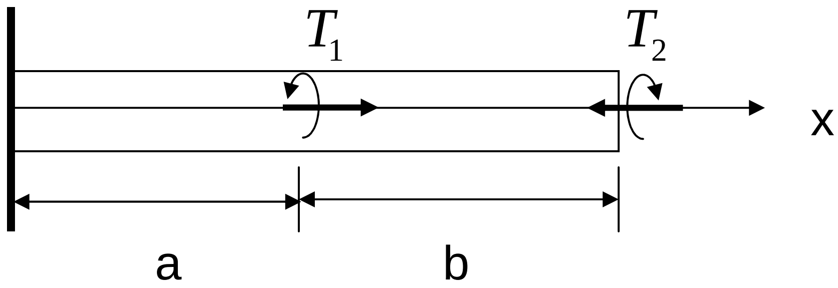
which is a maximum at the fixed end

$$\sigma_{xx}\Big|_{x=0} = -E\omega_p \frac{Tk}{GJ_{eff}} \tanh(kL)$$

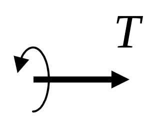
↑ varies along the section (ω_p)

Problem: Explain how $$\frac{d^2\beta}{dx^2} - k^2\beta = -k^2\frac{T}{GJ_{eff}}$$

can be used to solve this problem for $\phi = \phi(x)$:



0 < x < a $\quad T = T_1 - T_2$



$$\beta = \beta_1 = A\sinh(kx) + B\cosh(kx) + \frac{T_1 - T_2}{GJ_{eff}}$$

a < x < a + b $\quad T = -T_2$

$$\beta = \beta_2 = C\sinh(kx) + D\cosh(kx) - \frac{T_2}{GJ_{eff}}$$

To find A, B, C, D

$$\beta_1(0) = 0 \qquad (u_x = 0)$$

$$\frac{d\beta_2(a+b)}{dx} = 0 \qquad (\sigma_{xx} = 0)$$

$$\beta_1(a) = \beta_2(a) \qquad (\text{continuity of } u_x)$$

$$\frac{d\beta_1(a)}{dx} = \frac{d\beta_2(a)}{dx} \qquad (\text{continuity of } \sigma_{xx})$$

⇨ solve for A,B,C,D

in $0 < x < a$, integrate to get twist $\phi = \phi_1 = \int \beta_1 dx + C_1$

in $a < x < a + b$ integrate to get twist $\phi = \phi_2 = \int \beta_2 dx + C_2$

Find C_1 , C_2 from $\phi_1(0) = 0$ $\phi_1(a) = \phi_2(a)$