Restraint of Warping (open, thin sections)

\[ u_x = -\frac{d\phi}{dx} \omega \]

If \( d\phi/dx \) is no longer a constant, this will cause a normal stress to develop

\[ \sigma_{xx} = Ee_{xx} = E \frac{\partial u_x}{\partial x} \]

\[ = -E \omega \frac{d^2\phi}{dx^2} \]
this normal stress must not produce any axial force or bending moments

\[ \int_A \sigma_{xx} dA = E \phi'' \int_A \omega dA = 0 \]
\[ \int_A y \sigma_{xx} dA = E \phi'' \int_A y \omega dA = 0 \]
\[ \int_A z \sigma_{xx} dA = E \phi'' \int_A z \omega dA = 0 \]

which says that the sectorial area function must be a principal sectorial area function and we have

\[ \sigma_{xx} = -E \omega \frac{d^2 \phi}{dx^2} \]

Note: these stresses are self-equilibrated but they do not rapidly decay from the ends (thus, they violate Saint Venant's principle)
However, a changing axial stress will require that additional shear flows be developed (recall how we obtained the VQ/It expression for shear stress in bending)

\[ \sum F_x = 0 \]

\[
\left( \sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} \right) t ds - \sigma_{xx} t ds + \left( q(s) + \frac{\partial q}{\partial s} ds \right) dx - q(s) dx = 0
\]

\[
\frac{\partial q}{\partial s} = -t \frac{\partial \sigma_{xx}}{\partial x}
\]

\[
= E \frac{d^3 \phi}{dx^3} t \omega_p = E \frac{d^2 \beta}{dx^2} t \omega_p \quad \beta = \phi' = \frac{d \phi}{dx}
\]
integrating gives

\[ q(s) = E \frac{d^2 \beta}{dx^2} \int_0^s \omega_p \, td\!s \]

\[ = E \frac{d^2 \beta}{dx^2} \int \omega_p \, dA \]

which is the shear flow due to restraint of warping. The torque associated with this shear flow is

\[ T_q = \int_{s=0}^{s_{sf}} r_\perp q \, ds \]

\[ = \int_0^{s_f} q \, d\omega_p \]

Thus, the total torque is

\[ T = T_{SV} + T_q \]

\[ = GJ_{eff} \beta + \int_0^{s_f} q \, d\omega_p \]
Integration by parts lets us express this in the form

\[ T = GJ_{\text{eff}} \beta + q \omega_p \bigg|_{s = s_f}^{s = s_0} - \int_0^{s_f} \omega_p \left( \frac{\partial q}{\partial s} \right) ds \]

\[ = GJ_{\text{eff}} \beta - E \frac{d^2 \beta}{dx^2} \int_A \omega_p^2 dA \]

and defining \( J_\omega = \int_A \omega_p^2 dA \)

gives

\[ \frac{d^2 \beta}{dx^2} - k^2 \beta = -k^2 \frac{T}{GJ_{\text{eff}}} \]

\( k^2 = \frac{GJ_{\text{eff}}}{EJ_\omega} \)

differential equation for \( \beta(x) \)
If the torque $T = \text{constant}$, solution is

$$\beta = C \sinh(kx) + D \cosh(kx) + \frac{T}{GJ_{\text{eff}}}$$

$C$, $D$ are constants. We need to find these from the boundary conditions

Fixed end: $\beta = 0$ \hspace{1cm} ( $u_x = 0$ )

Unconstrained end: $\frac{d\beta}{dx} = 0$ \hspace{1cm} ( $\sigma_{xx} = 0$ )
\[ \beta|_{x=0} = 0 \quad \Rightarrow \quad \dot{D} = -\frac{T}{GJ_{\text{eff}}} \]

\[ \frac{d\beta}{dx}\bigg|_{x=L} = 0 \quad \Rightarrow \quad C = \frac{T}{GJ_{\text{eff}}} \tanh(kL) \]

which gives

\[ \beta = \frac{T}{GJ_{\text{eff}}} \left[ \tanh(kL)\sinh(kx) - \cosh(kx) + 1 \right] \]
\[
\beta = \frac{d\phi}{dx} = \frac{T}{GJ_{\text{eff}}} \left[ \tanh(kL) \sinh(kx) - \cosh(kx) + 1 \right]
\]

To get the twist we must integrate, using the additional boundary condition \( \phi(0) = 0 \). We obtain at \( x = L \)

\[
\phi(L) = \int_0^L \beta \, dx = \frac{TL}{GJ_{\text{eff}}} \left[ 1 - \frac{\tanh(kL)}{kL} \right]
\]

The axial stress in the bar is

\[
\sigma_{xx} = -E \omega_p \frac{d\beta}{dx} = -E \omega_p \frac{Tk}{GJ_{\text{eff}}} \left[ \tanh(kL) \cosh(kx) - \sinh(kx) \right]
\]

which is a maximum at the fixed end

\[
\sigma_{xx} \bigg|_{x=0} = -E \omega_p \frac{Tk}{GJ_{\text{eff}}} \tanh(kL)
\]

varies along the section
Problem: Explain how \[ \frac{d^2 \beta}{dx^2} - k^2 \beta = -k^2 \frac{T}{GJ_{\text{eff}}} \]
can be used to solve this problem for \( \phi = \phi(x) \):

\[ \begin{align*}
0 < x < a & \quad T = T_1 - T_2 \\
\beta &= \beta_1 = A \sinh (kx) + B \cosh (kx) + \frac{T_1 - T_2}{GJ_{\text{eff}}} \\
a < x < a + b & \quad T = -T_2 \\
\beta &= \beta_2 = C \sinh (kx) + D \cosh (kx) - \frac{T_2}{GJ_{\text{eff}}}
\end{align*} \]
To find A, B, C, D

\[ \beta_1(0) = 0 \quad (u_x = 0) \]

\[ \frac{d\beta_2(a + b)}{dx} = 0 \quad (\sigma_{xx} = 0) \]

\[ \beta_1(a) = \beta_2(a) \quad \text{(continuity of } u_x) \]

\[ \frac{d\beta_1(a)}{dx} = \frac{d\beta_2(a)}{dx} \quad \text{(continuity of } \sigma_{xx}) \]

in \( 0 < x < a \), integrate to get twist \( \phi = \phi_1 = \int \beta_1 dx + C_1 \)

in \( a < x < a + b \) integrate to get twist \( \phi = \phi_2 = \int \beta_2 dx + C_2 \)

Find \( C_1, C_2 \) from \( \phi_1(0) = 0 \quad \phi_1(a) = \phi_2(a) \)

solve for A, B, C, D