Prandtl Stress Function Summary

$$\begin{aligned} u_x &= u_x(y,z) \\ u_y &= -z\phi(x) \\ u_z &= y\phi(x) \end{aligned} \quad \Rightarrow \quad \begin{aligned} \sigma_{xz} &= G\left(\frac{\partial u_x}{\partial z} + \phi' y\right) \\ \sigma_{xy} &= G\left(\frac{\partial u_x}{\partial y} - \phi' z\right) \end{aligned} \qquad (1)$$

satisfy equilibrium equation $\dfrac{\partial \sigma_{xy}}{\partial y} + \dfrac{\partial \sigma_{xz}}{\partial z} = 0$

by taking

$$\begin{aligned} \sigma_{xy} &= G\phi' \frac{\partial \bar{\Phi}}{\partial z} \\ \sigma_{xz} &= -G\phi' \frac{\partial \bar{\Phi}}{\partial y} \end{aligned}$$

$\bar{\Phi}$ … Prandtl stresss function (l^2)

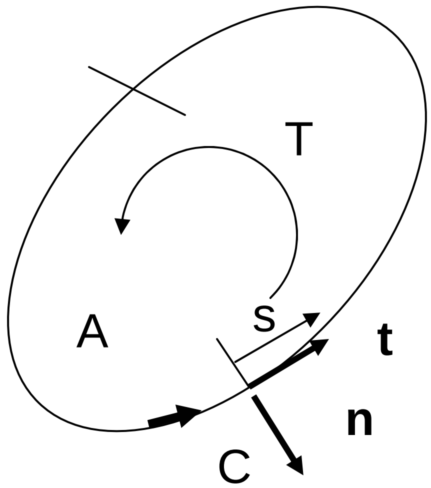
From (1) "compatibility"

$$G\frac{\partial^2 u_x}{\partial y \partial z} = -G\phi' \frac{\partial^2 \bar{\Phi}}{\partial y^2} - G\phi' = G\phi' \frac{\partial^2 \bar{\Phi}}{\partial z^2} + G\phi'$$

$$\Rightarrow \quad \frac{\partial^2 \bar{\Phi}}{\partial y^2} + \frac{\partial^2 \bar{\Phi}}{\partial z^2} = -2$$

Poisson’s equation

bar cross section



$$\frac{\partial^2\bar{\Phi}}{\partial y^2}+\frac{\partial^2\bar{\Phi}}{\partial z^2}=-2 \qquad \text{in A}$$

$$\sigma_{xn}=\sigma_{xy}n_y+\sigma_{xz}n_z=G\phi'\frac{d\bar{\Phi}}{ds}=0$$

$$\Rightarrow \quad \bar{\Phi}=\text{constant}=0 \qquad \text{on C}$$

Torque-twist

$$T=G\phi' J_{eff}$$

$$J_{eff}=2\iint \bar{\Phi}\,dA$$

$$\left(\sigma_{nt}\right)_{on\,C}=\frac{T}{J_{eff}}\left(-\frac{\partial\bar{\Phi}}{\partial n}\right)_{on\,C}$$

$$\left(\tau_{\max}\right)_{on\,C}=\frac{T}{J_{eff}}\left(-\frac{\partial\bar{\Phi}}{\partial n}\right)_{\max\ on\,C}$$

For the warping displacement

$$\frac{\partial u_x}{\partial z} = -\phi' y - \phi' \frac{\partial \bar{\Phi}}{\partial y}$$

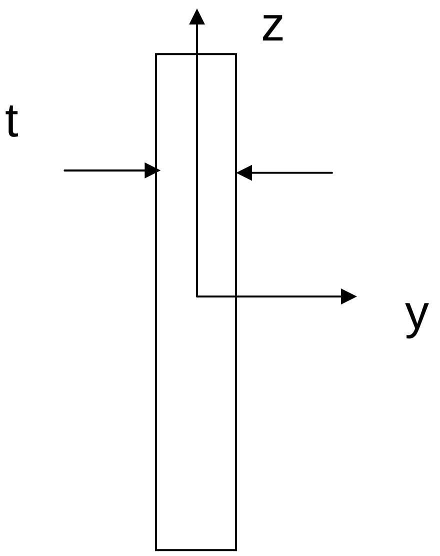
$$\frac{\partial u_x}{\partial y} = \phi' z + \phi' \frac{\partial \bar{\Phi}}{\partial z}$$

$$\Rightarrow$$

$$u_x = -\phi' \left(yz + \int \frac{\partial \bar{\Phi}}{\partial y} dz \right)$$

$$= \phi' \left(yz + \int \frac{\partial \bar{\Phi}}{\partial z} dy \right)$$

Thin rectangular cross section (neglect ends)



$$\bar{\Phi} = \left(\frac{t^2}{4} - y^2 \right) \qquad J_{eff} = 2 \int_{y=-t/2}^{y=+t/2} \bar{\Phi} b\, dy = \frac{1}{3} b t^3$$

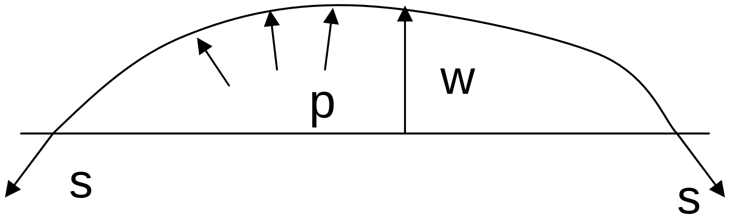
$$\tau_{\max} = \frac{T}{J_{eff}} \left(\mp \frac{\partial \bar{\Phi}}{\partial y} \right)_{y=\pm t/2} = \frac{3T}{bt^2}$$

$$u_x = \phi' yz$$

Membrane Analogy

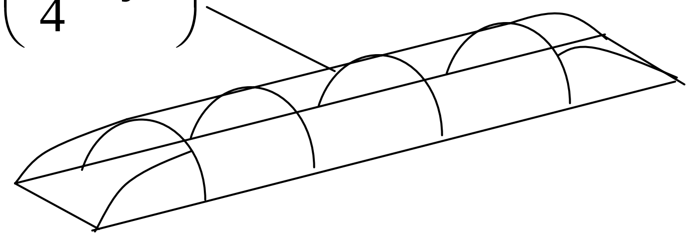
$$\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}=-\frac{p}{s}$$

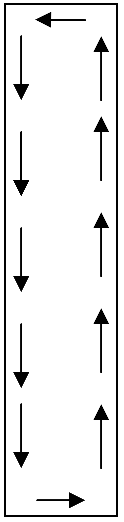
$$w=0$$ on the boundary



$$\bar{\Phi}=\frac{2s}{p}w$$

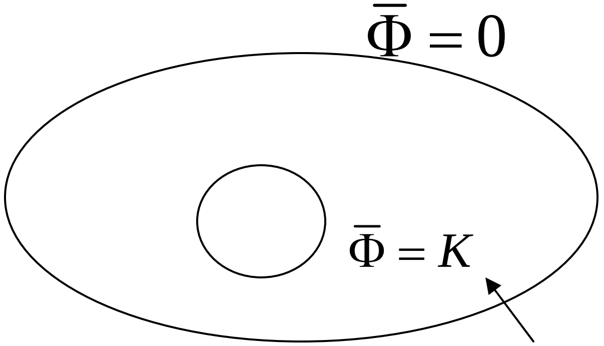
$$\bar{\Phi}=\left(\frac{t^2}{4}-y^2\right)$$





For cross sections with holes we also need to satisfy

$$\oint_{hole} du_x = 0$$



additional
unknown

$$\Rightarrow \quad \oint_{hole} \left(-\frac{\partial \bar{\Phi}}{\partial n} \right) ds = 2G\phi' A_{hole}$$

or

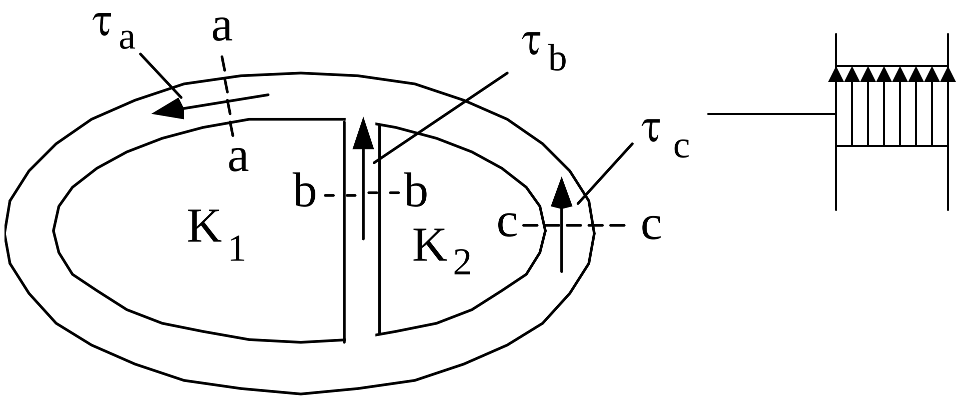
$$\oint_{hole} \tau ds = 2G\phi' A_{hole}$$

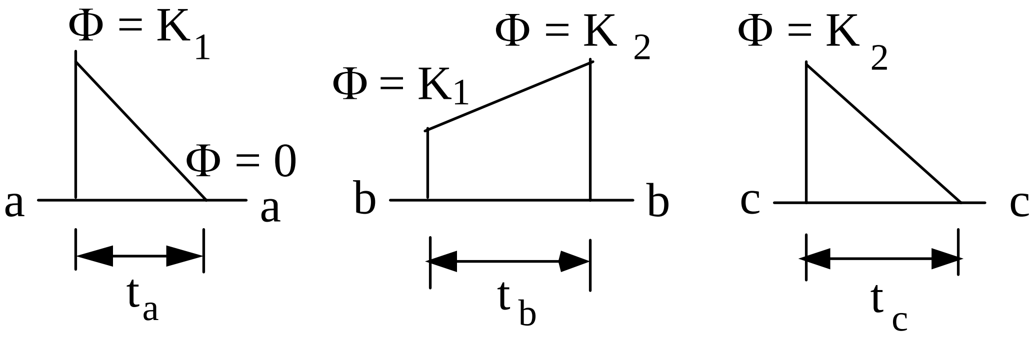
↑

supplementary condition to determine K

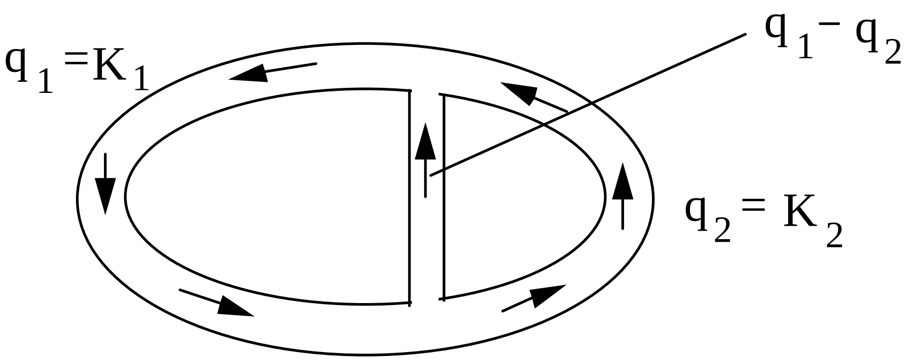
If one has multiple holes, this additional condition is applied at each hole to solve for the multiple unknown constants

Torsion of Thin, Closed Sections





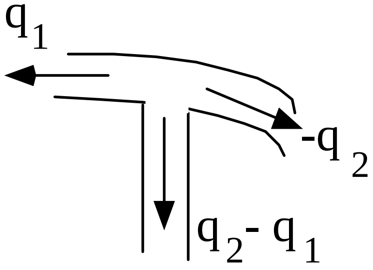
$$\tau_a = \frac{K_1}{t_a} \;, \quad \tau_b = \frac{K_1 - K_2}{t_b} \;, \quad \tau_c = \frac{K_2}{t_c}$$



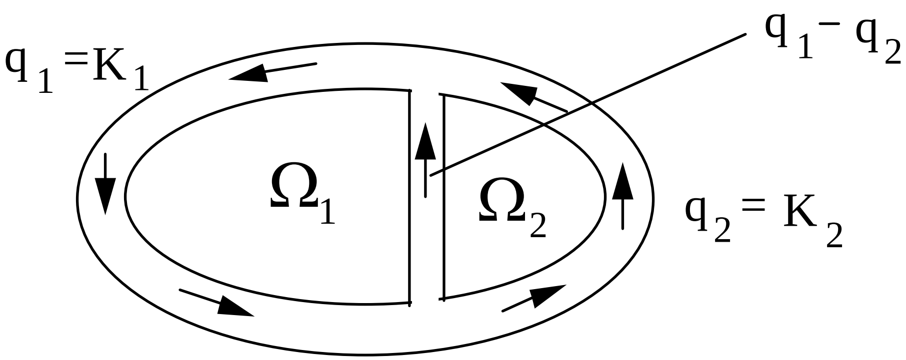
shear flows

$$q_1 = \tau_a t_a = K_1 \ , \quad q = q_1 - q_2 = \tau_b t_b = K_1 - K_2 \ , \quad q_2 = \tau_c t_c = K_2$$

shear flows into or out of a junction are conserved



$$\sum q_{out} = 0$$



Ω_i … area enclosed by centerline of ith "cell'

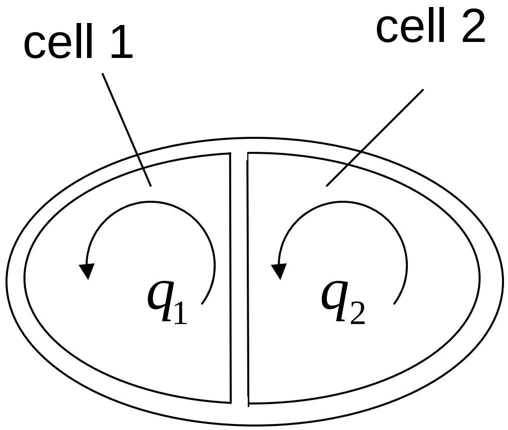
Torque-shear flow $$T = \sum_i 2 q_i \Omega_i$$

for each cell $$\oint_{hole} \tau\, ds = 2G\phi' A_{hole} \Rightarrow \frac{1}{2G\Omega_i} \oint_{ith\,cell} \frac{q}{t} ds = \phi'$$

$$\tau_{\max} = \left(K \frac{q}{t} \right)_{\max}$$

warping is generally small for closed sections

Torsion of a Thin Closed Section
(multiple cells)



$$T = 2\Omega_1 q_1 + 2\Omega_2 q_2 \qquad (1)$$

$$\phi' = \frac{1}{2G\Omega_1} \oint_{C_1} \frac{q}{t} ds \qquad (2)$$

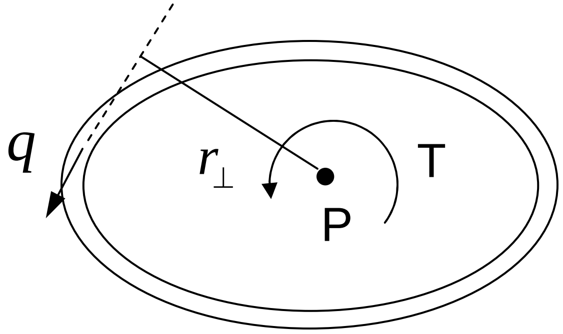
$$\phi' = \frac{1}{2G\Omega_2} \oint_{C_2} \frac{q}{t} ds \qquad (3)$$

(the q in Eqs.(2) and (3) is the <u>total q</u> flowing in a given cross section, i.e it is $q_1 - q_2$ flowing ↑ in the vertical section)

1. If the torque T is known, then q_1 and q_2 are first found in terms of the unknown ϕ' from Eqs. (2) and (3). These q_m 's are then placed into Eq.(1) which is solved for the unknown ϕ' . Once ϕ' is known in this manner, the q_m 's are completely determined.

2. If ϕ' is known, Eqs.(2) and (3) can be solved directly for the q_m 's and then Eq.(1) can be used to find the torque, T

For a single cell, we can write these more explicitly



Ω = area contained within the centerline of the cross section

$$T = \oint_C q r_\perp ds = q \oint_C r_\perp ds = 2q\Omega \quad \Rightarrow \quad \boxed{q = \frac{T}{2\Omega}}$$

$$\boxed{\tau_{\max} = \left(\frac{T}{2\Omega t}\right)_{\max} = \frac{T}{2\Omega t_{\min}}}$$

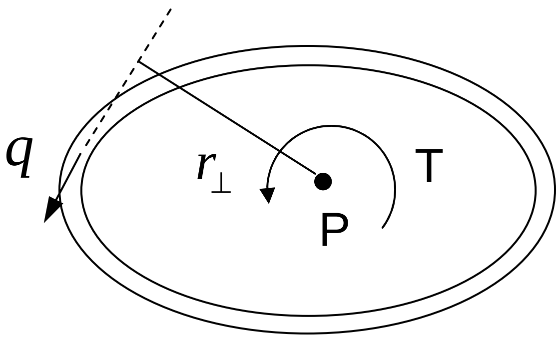
(no stress concentrations)

$$\phi' = \frac{1}{2G\Omega} \oint_C \frac{q}{t} ds = \frac{T}{4G\Omega^2} \oint_C \frac{ds}{t} \quad \Rightarrow \quad T = GJ_{eff}\phi'$$

where $\boxed{J_{eff} = \dfrac{4\Omega^2}{\oint_C \dfrac{ds}{t}}}$

Torsion of a Thin Closed Section
(single cell)

Ω = area contained within the centerline of the cross section



$$q = \frac{T}{2\Omega} \qquad (1)$$

$$T = GJ_{eff}\,\phi' \qquad (2)$$

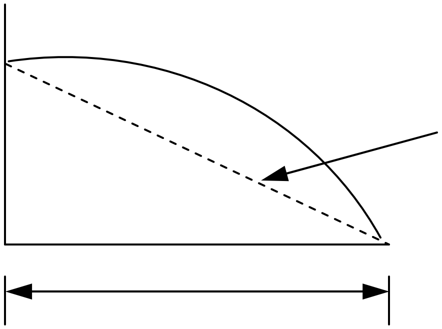
where $$J_{eff} = \frac{4\Omega^2}{\oint_C \frac{ds}{t}}$$

1. If T is known, q follows directly from Eq. (1), ϕ' is found from Eq.(2)

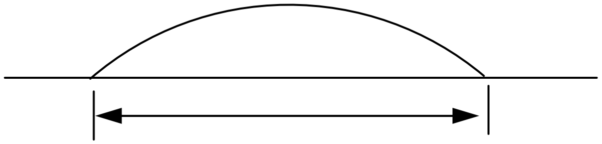
2. If ϕ' is known, T follows from Eq.(2), and q is then found from Eq. (1)

Torsion of a Thin Closed Section
(single cell)

The shear stress is not quite uniform across the thickness for thin closed sections



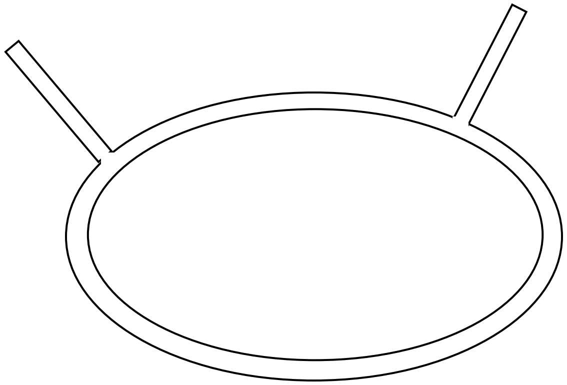
The difference looks much like that for an open section



so as a small correction factor:

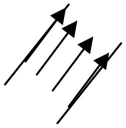
$$J_{eff} = \frac{4\Omega^2}{\oint_C \frac{ds}{t}} + \frac{1}{3}\oint_C t^3\left(s\right) ds$$

Torsion of closed sections with fins



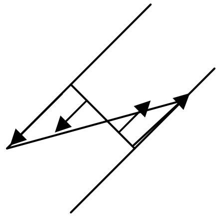
$$T = T_c + \sum T_f = G\phi'\left[\underbrace{\frac{4\Omega^2}{\oint \frac{ds}{t}} + \frac{1}{3}\oint t^3 ds}_{J_{eff}\text{ closed}} + \sum \underbrace{\frac{1}{3}\oint t^3 ds}_{J_{eff}\text{ for a fin (allows varaiable thickness)}}\right]$$

In the closed section $\tau_{\max} = \dfrac{T_c}{2\Omega t_{\min}}$ $T_c = G\phi'\left(J_{eff}\right)_{closed}$



In a fin

$$\tau_{\max} = \left(\frac{T_f t}{\left(J_{eff}\right)_{fin}} \right)_{\max}$$



$$T_f = G\phi' \left(J_{eff}\right)_{fin}$$

We can write this also in terms of the values since

$$J_{total} = J_f + J_{others}$$

$$T_{total} = T_f + T_{others}$$

$$T_f = G\phi' J_f$$

$$T_{others} = G\phi' J_{others}$$

$$T_{total} = G\phi' J_{total}$$

so

$$\frac{T_f}{\left(J_{eff}\right)_{fin}} = \frac{T_{total}}{\left(J_{eff}\right)_{total}} = G\phi'$$

$$\tau_{\max} = \left(\frac{T_{total} t}{\left(J_{eff}\right)_{total}} \right)_{\max}$$