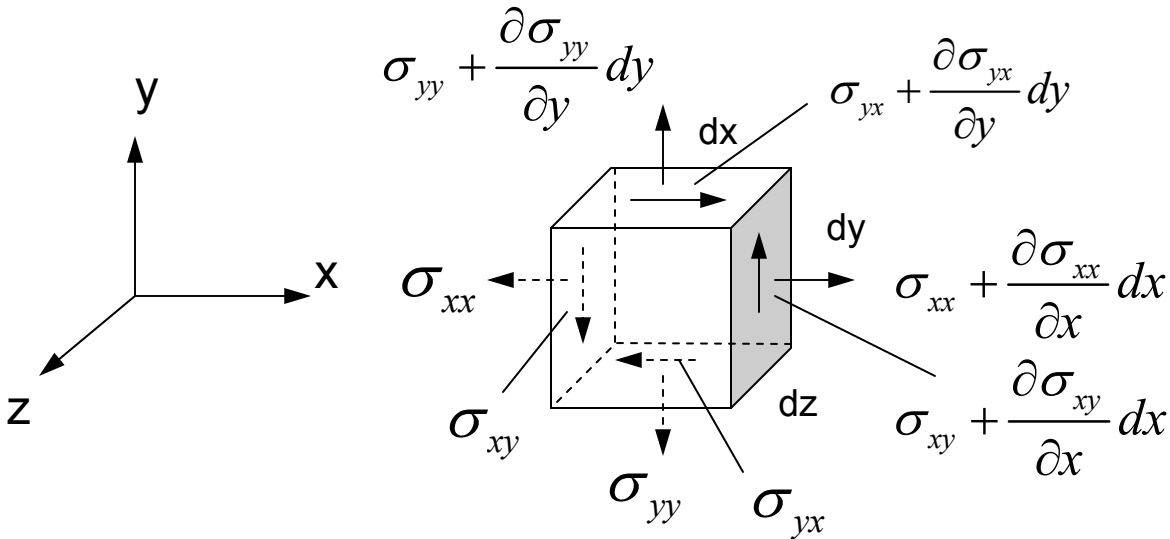


Local equilibrium (biaxial state of stress)



$$\sum F_x = 0$$

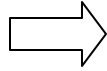
$$\left[\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx \right) dydz - \sigma_{xx} dydz \right] + \left[\left(\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y} dy \right) dx dz - \sigma_{yx} dx dz \right] = 0$$



$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0$$

similarly

$$\sum F_y = 0$$



$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

Equations of equilibrium
(2-D stresses)

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

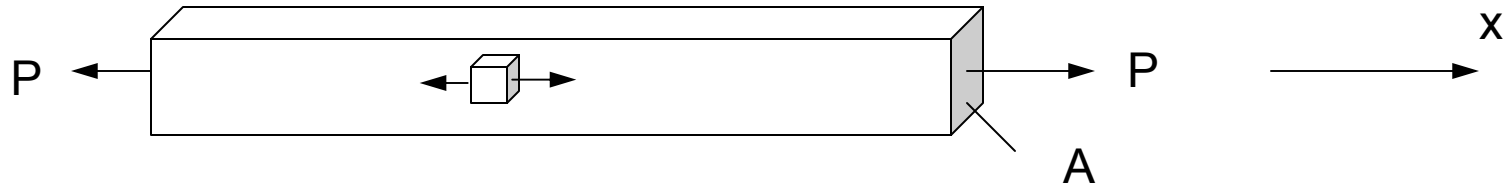
General 3-D Equations
of equilibrium (with
body forces)

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + f_x = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + f_y = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z = 0$$

Stresses from axial loading of a bar through the centroid of the cross sectional area



$$\sigma_{xx} = \frac{P}{A} = \text{constant}$$

all other stresses are zero

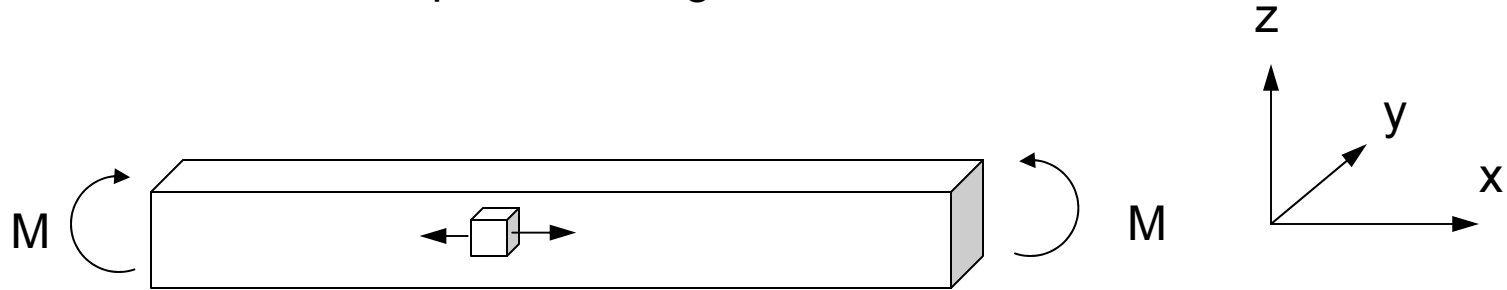
Equations of equilibrium are all satisfied (with zero body force)

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0$$

Stresses from pure bending of a beam



$$\sigma_{xx} = \frac{-M z}{I_{yy}}$$

all other stresses are zero

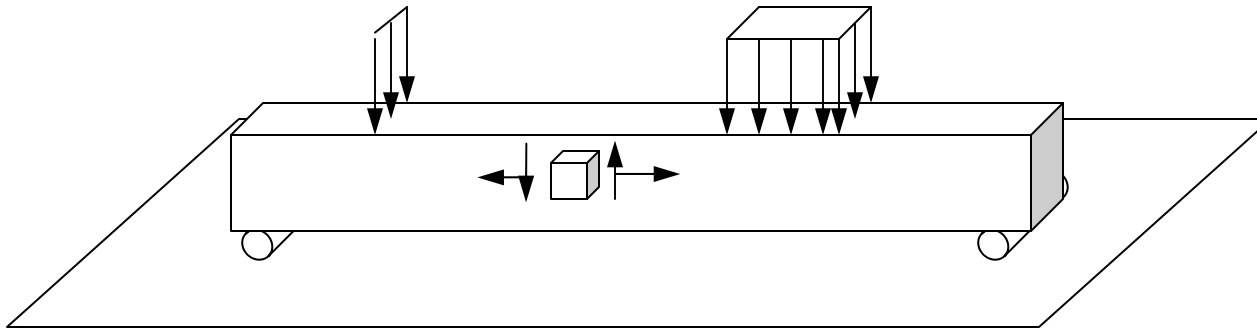
Equations of equilibrium are all satisfied (with zero body force)

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0$$

Stresses due to bending (engineering beam theory)



$$\sigma_{xx} = \frac{-M(x)z}{I_{yy}} \quad \sigma_{xz} = -\frac{V(x)Q(z)}{I_{yy}t(z)}$$

all other stresses are neglected

Are the equilibrium equations satisfied?

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0$$

Consider the equilibrium equations for a rectangular beam where $t(z) = t = \text{constant}$. Since there are no loads in the x and y directions it is reasonable to assume that $\sigma_{yy} = 0$, $\sigma_{xy} = 0$, $\sigma_{yz} = 0$, but the third equilibrium equation shows us that we cannot also set $\sigma_{zz} = 0$ and still satisfy that equilibrium equation

$$\frac{\partial}{\partial x} \left[\frac{-M(x)z}{I_{yy}} \right] + 0 + \frac{\partial}{\partial z} \left[\frac{-V(x)Q(z)}{I_{yy}t} \right] = 0$$

$$0 + 0 + 0 = 0$$

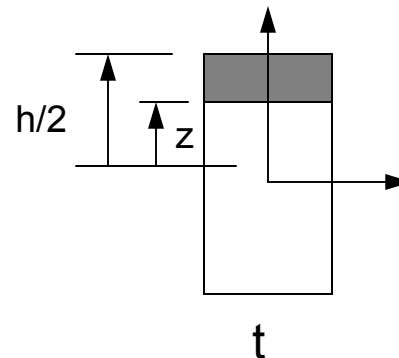
$$\frac{\partial}{\partial x} \left[\frac{-V(x)Q(z)}{I_{yy}t} \right] + 0 + \frac{\partial \sigma_{zz}}{\partial z} = 0$$

The first equilibrium equation is satisfied since

$$V(x) = \frac{dM(x)}{dx}$$

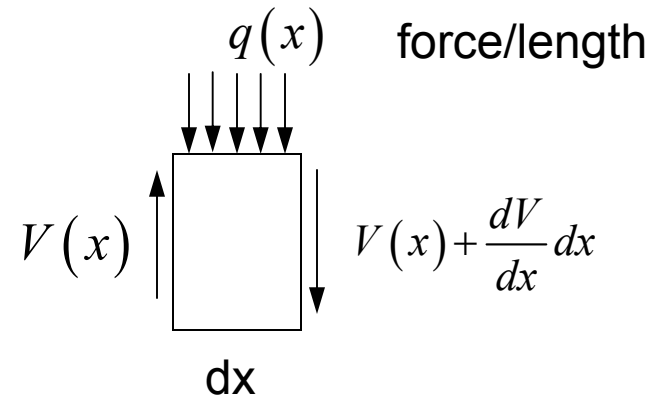
and

$$\frac{dQ(z)}{dz} = \frac{d}{dz} \left[\int_z^{h/2} z_0 t dz_0 \right] = -zt$$



From the third equilibrium equation

$$\begin{aligned}\frac{\partial \sigma_{zz}}{\partial z} &= \frac{dV/dx}{I_{yy}} \left[\frac{h^2}{8} - \frac{z^2}{2} \right] \\ &= \frac{-6q(x)}{th^3} \left[\frac{h^2}{4} - z^2 \right]\end{aligned}$$



$$\frac{dV}{dx} = -q(x)$$

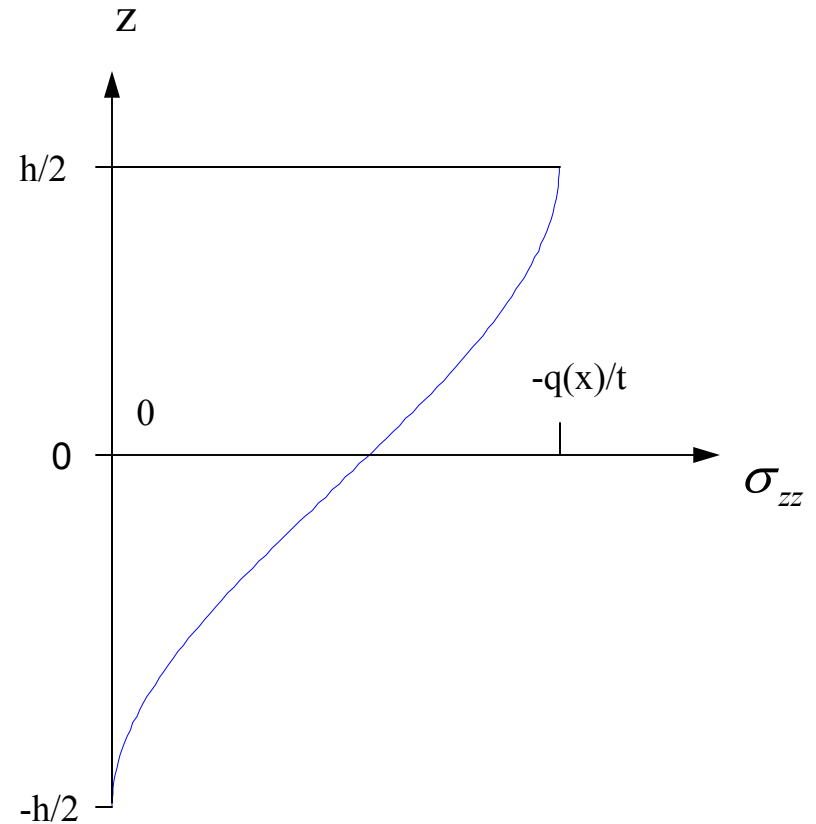
Integrating, we find

$$\sigma_{zz}(x, z) = \frac{-6q(x)}{th^3} \left[\frac{h^2 z}{4} - \frac{z^3}{3} + C \right]$$

and using $\sigma_{zz}(x, z = h/2) = -q(x)/t$ we find $C = \frac{h^3}{12}$

$$\sigma_{zz}(x, z) = \frac{-6q(x)}{th^3} \left[\frac{h^2 z}{4} - \frac{z^3}{3} + \frac{h^3}{12} \right]$$

$$\sigma_{zz}(x, z) = \frac{-6q(x)}{th^3} \left[\frac{h^2 z}{4} - \frac{z^3}{3} + \frac{h^3}{12} \right]$$



$$\left| \frac{(\sigma_{zz})_{\max}}{(\sigma_{xx})_{\max}} \right| = \frac{6q_{\max} L^2}{M_{\max}} \left(\frac{h}{L} \right)^2$$

\uparrow \uparrow
 O(1) very small
 for long,
 slender
 beams