Principle of Virtual Work and the Finite Element method

Consider a uniaxial load problem where a bar of variable cross sectional area carries a distributed body force, f_x, along its length as well as several discrete loads, P_a, P_b as shown The bar is fixed at its left end and we will let U_a, U_b be the displacement at the applied loads P_a, P_b, respectively. Then the principle of virtual work (or minimum potential energy) says that

$$\delta\Pi = \int \sigma_{xx}\delta e_{xx} dV - \int f_x \delta u_x dV - P_a\delta U_a - P_b\delta U_b = 0 \quad (1)$$

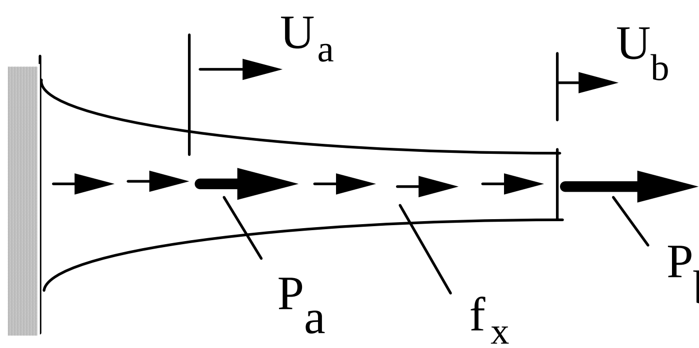


Fig.1

Now, suppose we break this bar into M elements and let l_m be the length of the mth element, U_m, U_{m+1} $(m = 1,...,M+1)$ be the end displacements of the mth element (called *nodal* displacements) , and F_m, F_{m+1} $(m = 1,...,M+1)$ be the “nodal” forces acting at the ends (nodes) of each element. For the problem shown above the force P_b at the right end acts on the (M+1)th node and we will assume that the elements are arranged so that the other force P_a acts at the pth node which has the nodal displacement U_p. For this problem there are only two non-zero applied nodal forces F_p and F_{M+1}, but for now we will continue to include all the other possible nodal forces as well.

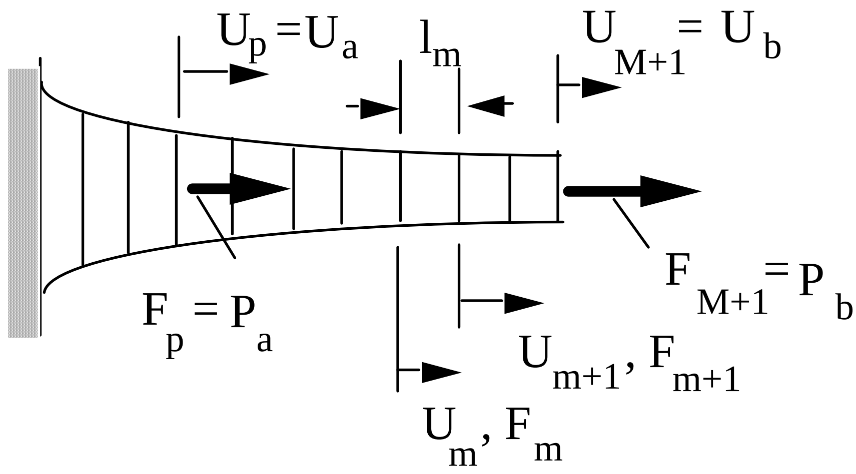


Fig.2

For the displacement in the *m*th element, we will assume that the displacement is a simple function which varies linearly between the two end (nodal) values U_m, U_{m+1}. This function is shown in Fig. 3.

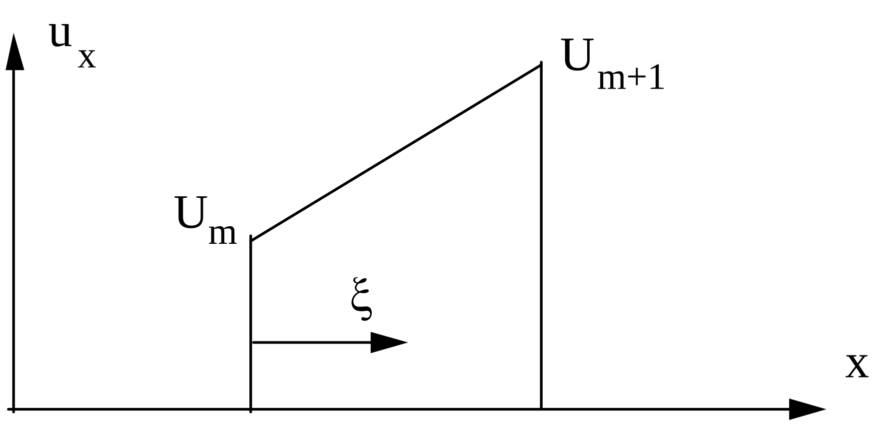


Fig. 3

Even though the displacement , $u_x^{(m)}$, in the *m*th element only depends on the nodal displacements at the end points of that element, we can include all the other nodal displacements as well by defining

$$u_x^{(m)} = \sum_{j=1}^{M+1} H_j^{(m)} U_j \quad (m = 1, \ldots, M) \tag{2}$$

where

$$H_j^{(m)} = \begin{cases} \left(1 - \dfrac{\xi}{l_m}\right) & j = m \\ \dfrac{\xi}{l_m} & j = m+1 \\ 0 & all\ other\ j \end{cases} \tag{3}$$

and where $0 < \xi < l_m$. The expression of Eq. (2) is equivalent to simply writing for the element

$$\begin{aligned} u_x^{(m)} &= U_m \left(1 - \frac{\xi}{l_m}\right) + U_{m+1}\left(\frac{\xi}{l_m}\right) \\ &= U_m + \frac{(U_{m+1} - U_m)\xi}{l_m} \end{aligned} \tag{4}$$

which is the displacement profile shown in Fig. 3. Using Eq. (2) the virtual displacement in the *m*th element can be written in terms of the virtual changes of all the nodal values as

$$\delta u_x^{(m)} = \sum_{j=1}^{M+1} H_j^{(m)} \delta U_j \tag{5}$$

By differentiation Eq. (2), we can also obtain the strain in the *m*th element in a similar form

$$e_{xx}^{(m)} = \frac{\partial u_x^{(m)}}{\partial \xi} = \sum_{j=1}^{M+1} J_j^{(m)} U_j \tag{6}$$

where

$$J_j^{(m)} = \begin{cases} -\dfrac{1}{l_m} & j = m \\ \dfrac{1}{l_m} & j = m+1 \\ 0 & all\ other\ j \end{cases} \tag{7}$$

which is just equivalent to expressing the strain as a constant in each element given by

$$e_{xx}^{(m)} = \frac{U_{m+1} - U_m}{l_m} \tag{8}$$

The virtual change of this strain, therefore, is also

$$\delta e_{xx}^{(m)} = \sum_{j=1}^{M+1} J_j^{(m)} \delta U_j \tag{9}$$

If we place all these results into the principle of virtual work for this problem (Eq. (1)) and use the fact that the stress is given by $\sigma_{xx} = E e_{xx}$ we obtain

$$\begin{aligned} &\sum_{m=1}^{M} \left\{ \int_{V_m} E \left[\sum_{k=1}^{M+1} J_k^{(m)} U_k \right] \left[\sum_{j=1}^{M+1} J_j^{(m)} \delta U_j \right] dV_m \right\} \\ &= \sum_{m=1}^{M} \left\{ \int_{V_m} f_x \left[\sum_{j=1}^{M+1} H_j^{(m)} \delta U_j \right] dV_m \right\} + \sum_{j=1}^{M+1} F_j d_j \delta U_j \end{aligned} \tag{10}$$

where the sum over *m* is the result of adding up the distributed work-energy terms over all the elements and the constants d_j are defined to eliminate all the nodal forces F_j except for those where applied forces act, e.g. for our case

$$d_j = \begin{cases} 1 & for\ j = p,\ F_p = P_a \\ 1 & for\ j = M+1, F_{M+1} = P_b \\ 0 & for\ all\ other\ j \end{cases} \tag{11}$$

Note that strictly speaking since we allowed the nodal displacement U_1 to vary here (we have kept δU_1) , the reaction force at that node should be included in the non-zero F_j terms since that reaction would do work, just like the applied loads. But since later we will enforce the boundary condition $U_1 = 0$, we can omit this reaction force in anticipation of the fact that it will not enter later since it does no virtual work.

Interchanging the orders of summation in Eq. (10) and combining all the terms then gives

$$\sum_{j=1}^{M+1} \delta U_j \left\{ \sum_{m=1}^{M} \int_{V_m} E \sum_{k=1}^{M+1} J_j^{(m)} J_k^{(m)} U_k dV_m - \sum_{m=1}^{M} \int_{V_m} f_x H_j^{(m)} dV_m + F_j d_j \right\} = 0 \tag{12}$$

However, since we assume the δU_j are all arbitrary, each of the terms multiplying these virtual displacement variations must vanish, to give

$$\sum_{m=1}^{M}\left\{\int_{V_m} E\sum_{k=1}^{M+1} J_j^{(m)} J_k^{(m)} U_k dV_m\right\} = \sum_{m=1}^{M}\left\{\int_{V_m} f_x H_j^{(m)} dV_m\right\} + F_j d_j \tag{13}$$

which is just a set of linear equations for the unknown nodal displacements of the form

$$\sum_{k=1}^{M+1} K_{jk} U_j = R_j + F_j d_j \tag{14}$$

where the *stiffness matrix*, *K*, is given by

$$K_{jk} = \sum_{m=1}^{M}\left\{\int_{V_m} E J_j^{(m)} J_k^{(m)} dV_m\right\} \tag{15}$$

and the body force term is

$$R_j = \sum_{m=1}^{M}\left\{\int_{V_m} f_x H_j^{(m)} dV_m\right\} \tag{16}$$

We can write Eq. (14) out in matrix form as

$$\begin{bmatrix} K_{11} & \dots & \dots & K_{1M+1} \\ \dots & & & \dots \\ & & & \\ \dots & & & \dots \\ K_{M+11} & \dots & \dots & K_{M+1M+1} \end{bmatrix} \begin{Bmatrix} U_1 \\ \dots \\ \\ \dots \\ U_{M+1} \end{Bmatrix} = \begin{Bmatrix} R_1 \\ \dots \\ \\ \dots \\ R_{M+1} \end{Bmatrix} + \begin{Bmatrix} F_1 d_1 \\ \dots \\ \\ \dots \\ F_{M+1} d_{M+1} \end{Bmatrix} \tag{17}$$

Before this system of equations can be solved, we must impose the displacement boundary condition $U_1 = 0$. This can be done by simply eliminating the first equation (since $\delta U_1 = 0$ this first equation should not have appeared in the first place anyway) and forcing all the terms involving U_1 to be zero. This is equivalent to zeroing out the first row and column of terms in Eq. (17) to obtain

$$\begin{bmatrix} 0 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & K_{22} & \ldots & & & K_{2M+1} \\ \ldots & & & & & \ldots \\ \ldots & & & & & \\ 0 & & & & & \ldots \\ 0 & K_{M+12} & \ldots & & \ldots & K_{M+1M+1} \end{bmatrix} \begin{Bmatrix} 0 \\ U_2 \\ \ldots \\ \\ \ldots \\ U_{M+1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ R_2 \\ \ldots \\ \\ \ldots \\ R_{M+1} \end{Bmatrix} + \begin{Bmatrix} 0 \\ F_2 d_2 \\ \ldots \\ \\ \ldots \\ F_{M+1} d_{M+1} \end{Bmatrix} \quad (18)$$

which is system of *M* equations in the remaining *M* unknown nodal displacements that can then be solved. Once all the nodal displacements are found, then Eqs. (2) and (6), in conjunction with Hooke's law, can be used to find the displacements, strains, and stresses throughout the bar.

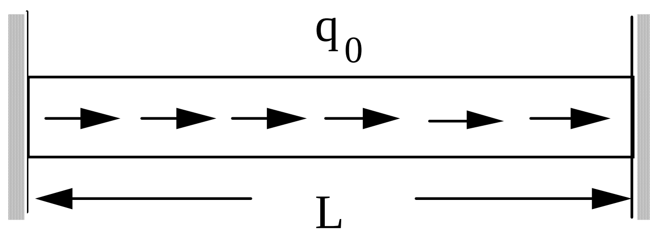
Note that if some other displacement was specified in the bar as a non-zero value, then that constraint could also be handled in much the same way as the boundary condition $U_1 = 0$. For example, if the displacement at the third node was specified to be $U_3 = \tilde{U} \neq 0$, then this condition could be incorporated by zeroing out all the terms in the third row of Eq. (18) and moving the third column terms involving U_3 to the right side of the equations as known terms, i.e.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & K_{22} & 0 & K_{24} & \ldots & K_{2\,M+1} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & K_{42} & 0 & K_{44} & \ldots & K_{4\,M+1} \\ \ldots & \ldots & \ldots & \ldots & & \ldots \\ 0 & K_{M+1\,2} & 0 & K_{M+14} & \ldots & K_{M+1M+1} \end{bmatrix} \begin{Bmatrix} 0 \\ U_2 \\ 0 \\ U_4 \\ \ldots \\ U_{M+1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ R_2 \\ 0 \\ R_4 \\ \ldots \\ R_{M+1} \end{Bmatrix} + \begin{Bmatrix} 0 \\ F_2 d_2 \\ 0 \\ F_4 d_4 \\ \ldots \\ F_{M+1} d_{M+1} \end{Bmatrix} - \begin{Bmatrix} 0 \\ K_{23}\tilde{U} \\ 0 \\ K_{43}\tilde{U} \\ \ldots \\ K_{M+13}\tilde{U} \end{Bmatrix} \quad (19)$$

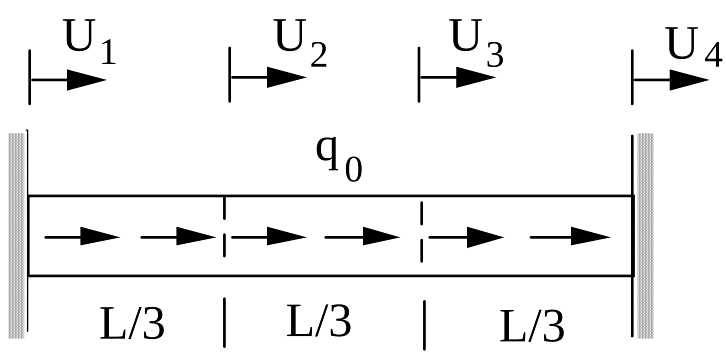
to give a resulting system of *M-1* equations to be solved for the remaining *M-1* unknowns.

Example

Consider the problem of a bar of constant cross-sectional area A and Young's modulus E loaded by uniform distributed load q_0 lb/unit length $(f_x = \frac{q_0}{A})$. By breaking this bar into three equal length finite elements, solve for the nodal displacements, assuming that the strains are constant in each element. Compare the predictions for displacement and stress with the exact solution for this problem.



The three elements contain four nodes labeled as shown below. In forming up the stiffness matrix elements, it is convenient to consider the individual contributions first from each element and then sum over all elements. Thus, we will write each element



contribution as

$$K_{jk}^{(m)} = \int_{V_m} E J_j^{(m)} J_k^{(m)} dV_m \tag{20}$$

so that the total stiffness matrix is

$$K_{jk} = \sum_{m=1}^{M} K_{jk}^{(m)} \tag{21}$$

Similarly, for the body force

$$R_j^{(m)} = \int_{V_m} f_x H_j^{(m)} dV_m$$
$$R_j = \sum_{m=1}^{M} R_j^{(m)} \tag{22}$$

Then, for the first element, taking its length as l ($l = L/3$) we have

$$K_{11}^{(1)} = EA\int_0^l J_1^{(1)} J_1^{(1)} d\xi = EA\int_0^l \left(\frac{-1}{l}\right)\left(\frac{-1}{l}\right) d\xi = \frac{EA}{l}$$
$$K_{12}^{(1)} = EA\int_0^l J_1^{(1)} J_2^{(1)} d\xi = EA\int_0^l \left(\frac{-1}{l}\right)\left(\frac{1}{l}\right) d\xi = \frac{-EA}{l}$$
$$K_{21}^{(1)} = EA\int_0^l J_2^{(1)} J_1^{(1)} d\xi = K_{12}^{(1)} \tag{23}$$
$$K_{22}^{(1)} = EA\int_0^l J_2^{(1)} J_2^{(1)} d\xi = EA\int_0^l \left(\frac{1}{l}\right)\left(\frac{1}{l}\right) d\xi = \frac{EA}{l}$$
$$\textit{all other } K_{jk} = 0$$

and for the distributed load on this element

$$R_1^{(1)} = q_0\int_0^l H_1^{(1)} d\xi = q_0\int_0^l \left(1 - \frac{\xi}{l}\right) d\xi = \frac{q_0 l}{2}$$
$$R_2^{(1)} = q_0\int_0^l H_2^{(1)} d\xi = q_0\int_0^l \left(\frac{\xi}{l}\right) d\xi = \frac{q_0 l}{2} \tag{240}$$
$$\textit{all other } R_j^{(1)} = 0$$

Since the calculations for the other two elements involve similar evaluations, we only quote the end results below

For the second element

$$K_{22}^{(2)} = K_{33}^{(2)} = \frac{EA}{l}$$
$$K_{23}^{(2)} = K_{32}^{(2)} = -\frac{EA}{l}$$
$$\textit{all other } K_{jk}^{(2)} = 0 \tag{25}$$
$$R_2^{(2)} = R_3^{(2)} = \frac{q_0 l}{2}$$
$$\textit{all other } R_j^{(2)} = 0$$

and for the third element

$$
\begin{aligned}
&K_{33}^{(3)} = K_{44}^{(3)} = \frac{EA}{l} \\
&K_{34}^{(3)} = K_{43}^{(3)} = -\frac{EA}{l} \\
&\text{all other } K_{jk}^{(3)} = 0 \\
&R_3^{(3)} = R_4^{(3)} = \frac{q_0 l}{2} \\
&\text{all other } R_j^{(3)} = 0
\end{aligned} \tag{26}
$$

If we add up all these element contributions then the set of simultaneous equations has the form

$$
\begin{bmatrix}
\frac{EA}{l} & -\frac{EA}{l} & 0 & 0 \\
-\frac{EA}{l} & \frac{EA}{l}+\frac{EA}{l} & -\frac{EA}{l} & 0 \\
0 & -\frac{EA}{l} & \frac{EA}{l}+\frac{EA}{l} & -\frac{EA}{l} \\
0 & 0 & -\frac{EA}{l} & \frac{EA}{l}
\end{bmatrix}
\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}
=
\begin{Bmatrix} \frac{q_0 l}{2} \\ \frac{q_0 l}{2}+\frac{q_0 l}{2} \\ \frac{q_0 l}{2}+\frac{q_0 l}{2} \\ \frac{q_0 l}{2} \end{Bmatrix} \tag{27}
$$

where we have shown how all of the element contributions add together to give the final result. Applying the boundary conditions $U_1 = U_4 = 0$ means that we must cross out the first and fourth rows and columns and solve the remaining two equations in two unknowns given by

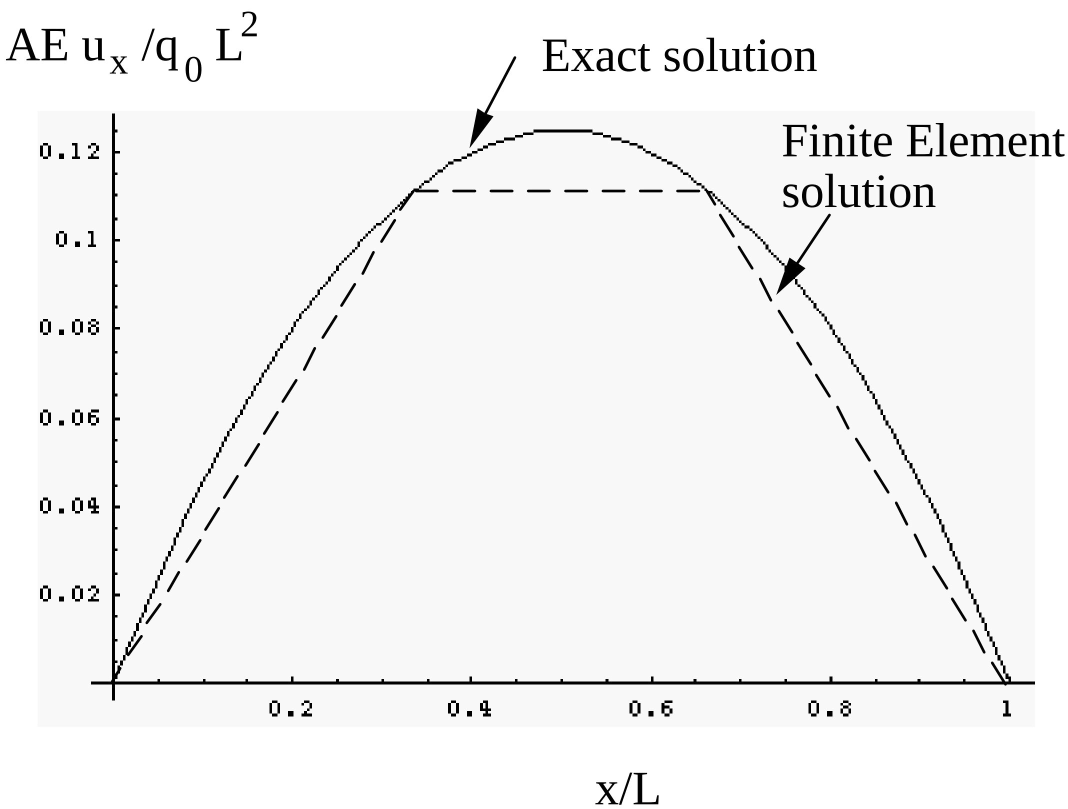
$$
\begin{bmatrix}
\frac{2AE}{l} & -\frac{AE}{l} \\
-\frac{AE}{l} & \frac{2AE}{l}
\end{bmatrix}
\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix}
=
\begin{Bmatrix} q_0 l \\ q_0 l \end{Bmatrix} \tag{28}
$$

which has the solution $U_2 = U_3 = \dfrac{q_0 l^2}{AE} = \dfrac{q_0 L^2}{9AE}$.

We can compare these results with the exact solution for this problem which is easy to obtain. The reader should verify that the exact solution for the displacement function and its specific values at the nodal locations are given by

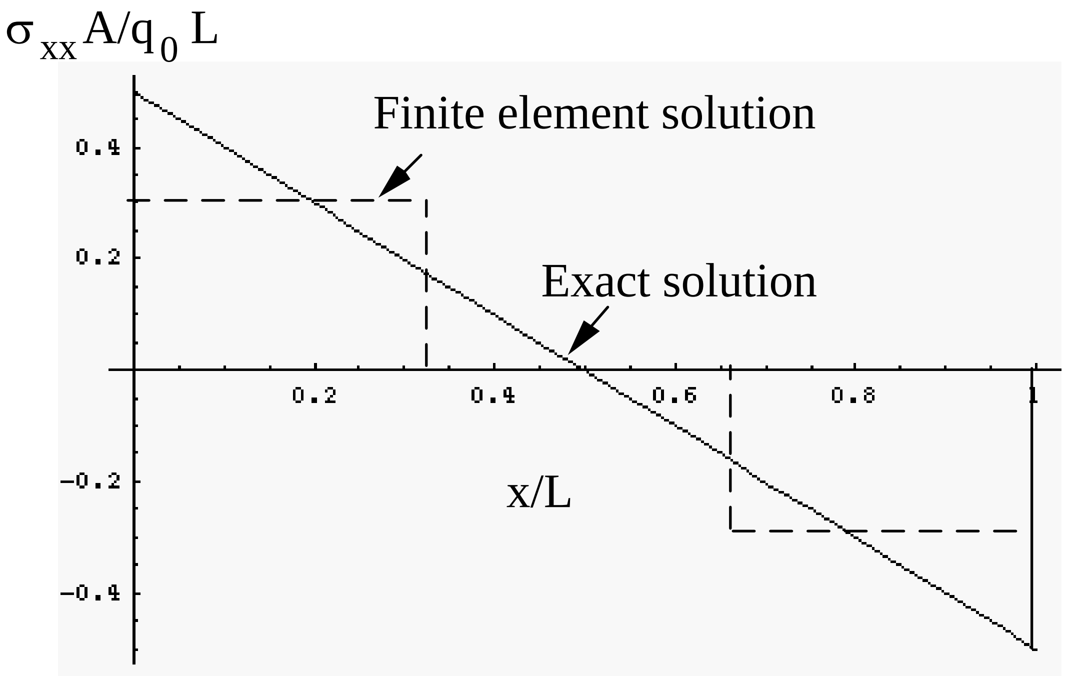
$$u_x(x) = \frac{q_0 L x}{2AE} - \frac{q_0 x^2}{2AE}$$
$$u_x\left(\frac{L}{3}\right) = u_x\left(\frac{2L}{3}\right) = \frac{q_0 L^2}{9AE} \tag{29}$$

so that the finite element solution values at the nodes are, in fact, exact.



The exact behavior of the axial stress for this problem also is easily obtained as

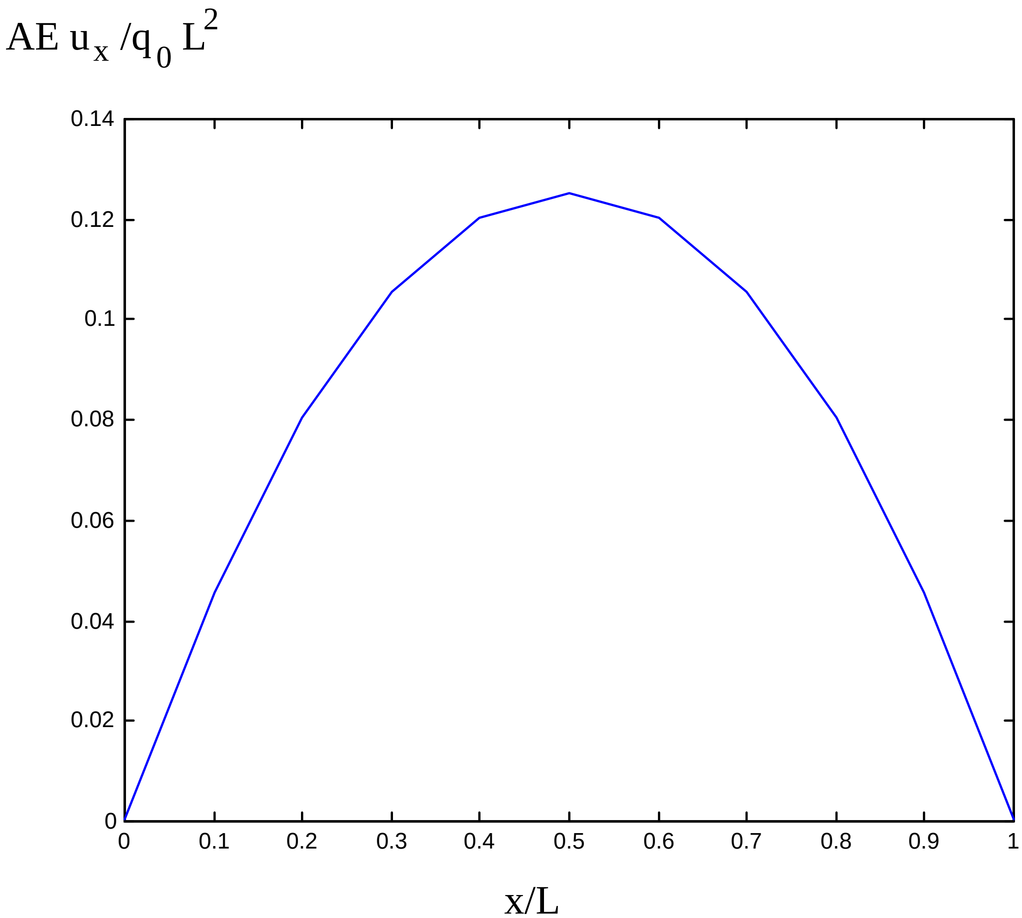
$$\sigma_{xx} = \frac{q_0 L}{2A} - \frac{q_0 x}{A} \tag{30}$$



It is obvious that the finite element solution does a better job of representing the displacements than the stresses. This is to be expected since the stresses involve derivatives of the approximating displacement functions.

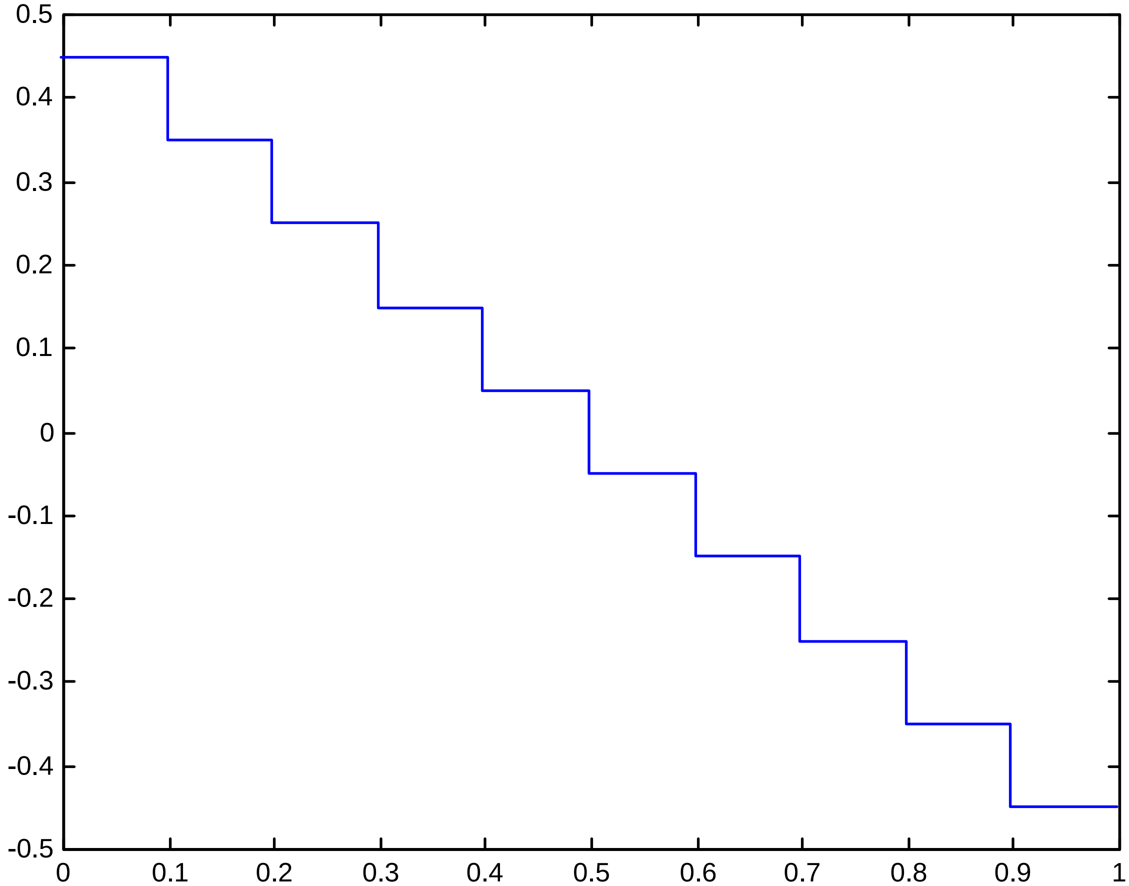
Finite Element Solution for 10 elements:

If we simply increase the number of elements to ten, the solution for the displacement becomes much closer to the exact solution



and the same is true for the stresses:

$\sigma_{xx} A/q_0 L$



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