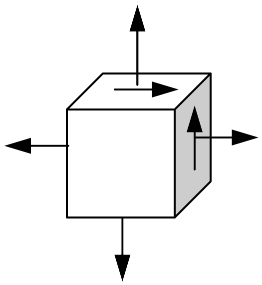
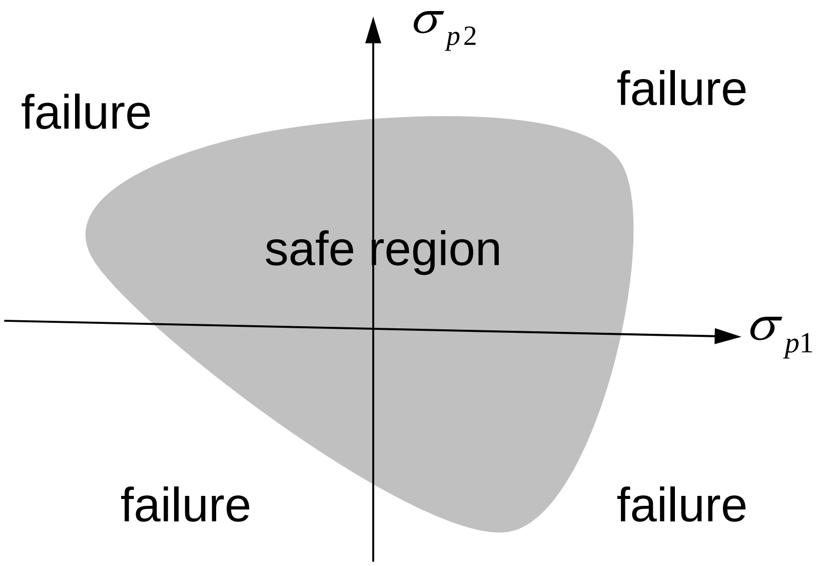
Theories of (static) Failure

Consider biaxial (plane stress) case first:

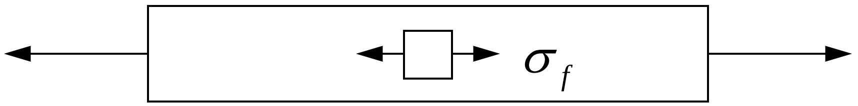


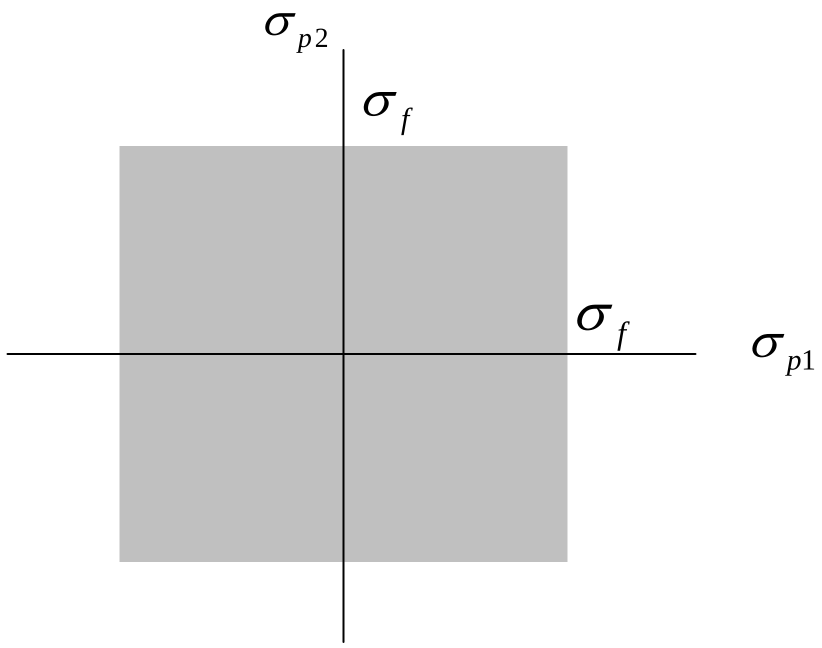
Basic idea is that if some combination of the principal stresses gets too large, the material will fail. We can think of the "safe" stresses as defining some region in terms of principal stress components:

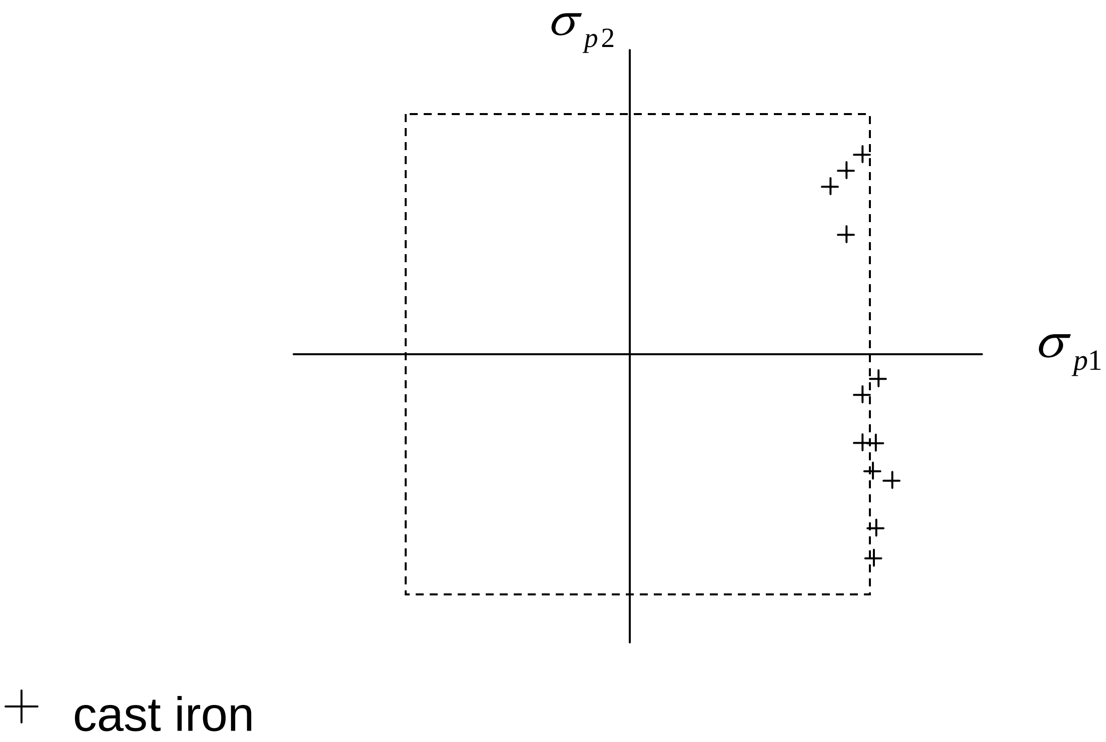


Maximum Normal Stress Theory

Failure occurs if either $|\sigma_{p1}|$ or $|\sigma_{p2}|$ = σ_f where σ_f is the fracture stress as determined in a uniaxial tension test






σ_{p2}
σ_{p1}
+ cast iron

Maximum Normal Stress Theory

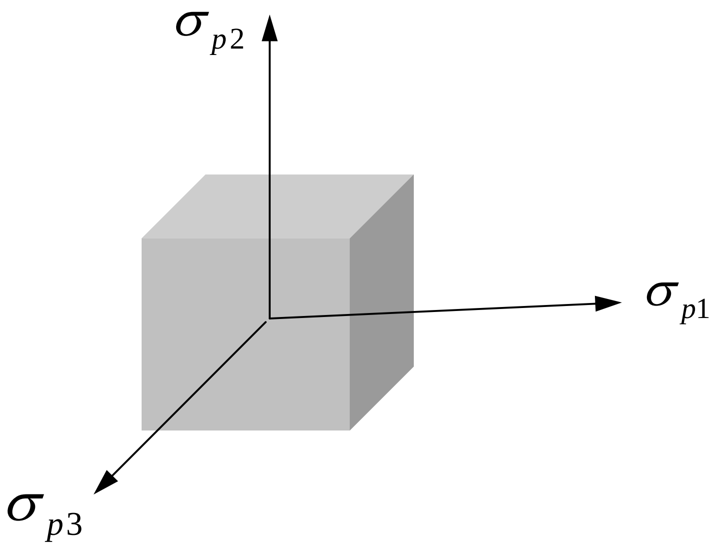
1. Used to describe fracture of brittle materials such as cast iron

2. Limitations

   doesn't distinguish between tension or compression

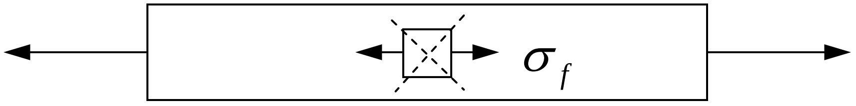
   doesn't depend on orientation of principal planes so only applicable to isotropic materials

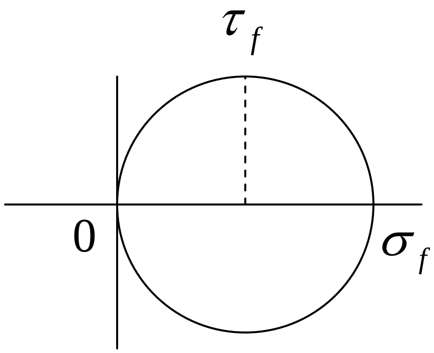
3. Generalization to 3-D stress case is easy:



Maximum Shearing Stress (Tresca) Theory

Failure by slip (yielding) occurs when the maximum shearing stress, τ_{max} exceeds the yield stress τ_f as determined in a uniaxial tension test



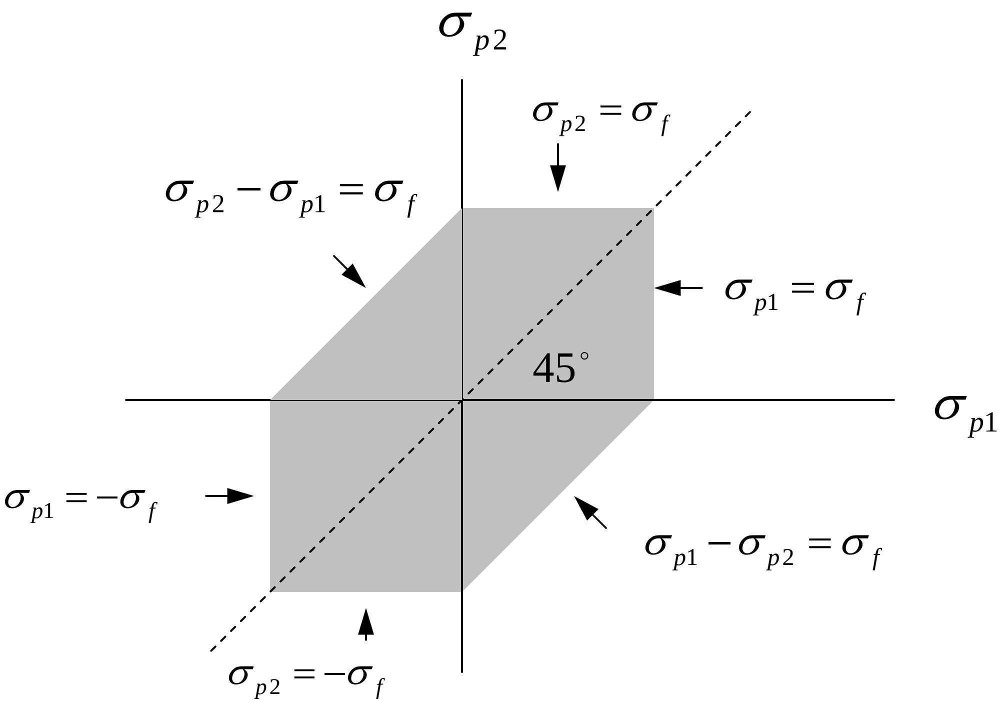


For biaxial (plane stress) case

$$\tau_{\max} = \tau_f = \sigma_f / 2 = \max \begin{cases} \left|\sigma_{p1} - \sigma_{p2}\right| / 2 \\ \left|\sigma_{p1}\right| / 2 \\ \left|\sigma_{p2}\right| / 2 \end{cases}$$

$$\tau_{\max} = \tau_f = \sigma_f / 2 = \max \begin{cases} \left|\sigma_{p1} - \sigma_{p2}\right| / 2 \\ \left|\sigma_{p1}\right| / 2 \\ \left|\sigma_{p2}\right| / 2 \end{cases}$$

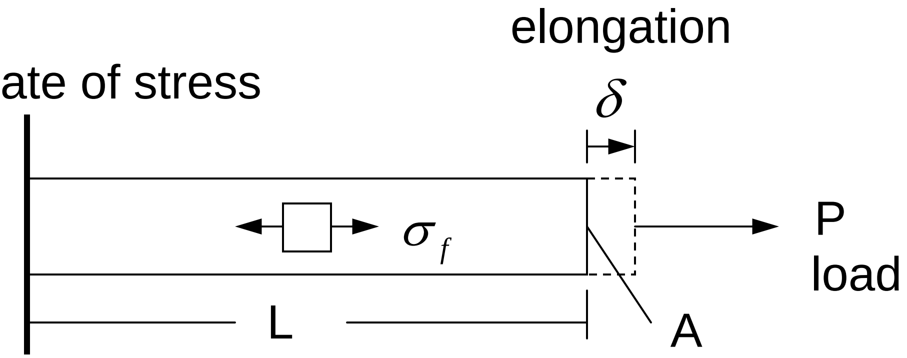
defines a hexagon in terms of the principal stresses



1. Limitation – doesn't depend on orientations of planes of extreme shear so only strictly applicable to isotropic materials

Maximum Distortional Strain Energy (von Mises) Theory

For 1-D state of stress



axial strain

$$e = \frac{\delta}{L}$$

Work done by P = strain energy (potential energy) of the bar

$$W_k = U$$

$$\begin{aligned} W_k &= \int P d\delta \\ &= \int \sigma A d\left[eL\right] \\ &= \underbrace{AL}\int \sigma de \end{aligned}$$

so that $U = AL\int \sigma de$ and

$AL = V$ = volume of bar

$$u = \frac{U}{V} = \int \sigma de$$

= strain energy/unit volume
(strain energy density)

Maximum Distortional Strain Energy (von Mises) Theory

$$u = \int \sigma de$$

But $\sigma = Ee$ so $u = E\int e\, de$

$$= \frac{Ee^2}{2} = \frac{\sigma e}{2}$$

For a general 3-D state of stress with three principal stresses

$$u = \frac{1}{2}\left(\sigma_{p1}e_{p1} + \sigma_{p2}e_{p2} + \sigma_{p3}e_{p3}\right)$$

Maximum Distortional Strain Energy (von Mises) Theory

$$u = \frac{1}{2}\left(\sigma_{p1}e_{p1} + \sigma_{p2}e_{p2} + \sigma_{p3}e_{p3}\right)$$

By generalized Hooke's law

$$e_{p1} = \frac{1}{E}\left[\sigma_{p1} - \nu\left(\sigma_{p2} + \sigma_{p3}\right)\right]$$

$$e_{p2} = \frac{1}{E}\left[\sigma_{p2} - \nu\left(\sigma_{p1} + \sigma_{p3}\right)\right]$$

$$e_{p3} = \frac{1}{E}\left[\sigma_{p3} - \nu\left(\sigma_{p1} + \sigma_{p2}\right)\right]$$

so

$$u = \frac{1}{2E}\left[\sigma_{p1}^2 + \sigma_{p2}^2 + \sigma_{p3}^2 - 2\nu\left(\sigma_{p1}\sigma_{p2} + \sigma_{p1}\sigma_{p3} + \sigma_{p2}\sigma_{p3}\right)\right]$$

Maximum Distortional Strain Energy (von Mises) Theory

$$u = \frac{1}{2E}\left[\sigma_{p1}^2 + \sigma_{p2}^2 + \sigma_{p3}^2 - 2\nu\left(\sigma_{p1}\sigma_{p2} + \sigma_{p1}\sigma_{p3} + \sigma_{p2}\sigma_{p3}\right)\right]$$

Experiments have shown that a pure hydrostatic pressure will not cause yielding even under extremely large stresses so that failure by slip must be independent of the hydrostatic part of u.

Let $\sigma_{p1} = \sigma_{p2} = \sigma_{p3} = -p$

Then $u_p = \frac{1}{2E}\left[3p^2 - 6\nu p^2\right] = -\frac{3(1-2\nu)}{2E}p^2$

and if we let $p = \frac{-\left(\sigma_{p1} + \sigma_{p2} + \sigma_{p3}\right)}{3}$

we have $u_p = \frac{(1-2\nu)}{6E}\left(\sigma_{p1} + \sigma_{p2} + \sigma_{p3}\right)^2$

Maximum Distortional Strain Energy (von Mises) Theory

Thus if we define the distortional strain energy density , u_d , as

$$u_d = u - u_p$$

it can be shown that

$$u_d = \frac{(1+\nu)}{6E}\left[\left(\sigma_{p1}-\sigma_{p2}\right)^2+\left(\sigma_{p1}-\sigma_{p3}\right)^2+\left(\sigma_{p2}-\sigma_{p3}\right)^2\right]$$

The von Mises failure theory predicts failure with respect to slip (yielding) whenever u_d equals u_f as determined by a uniaxial tension test

$$u_f = \frac{(1+\nu)}{6E}\left[2\sigma_f^2\right]$$

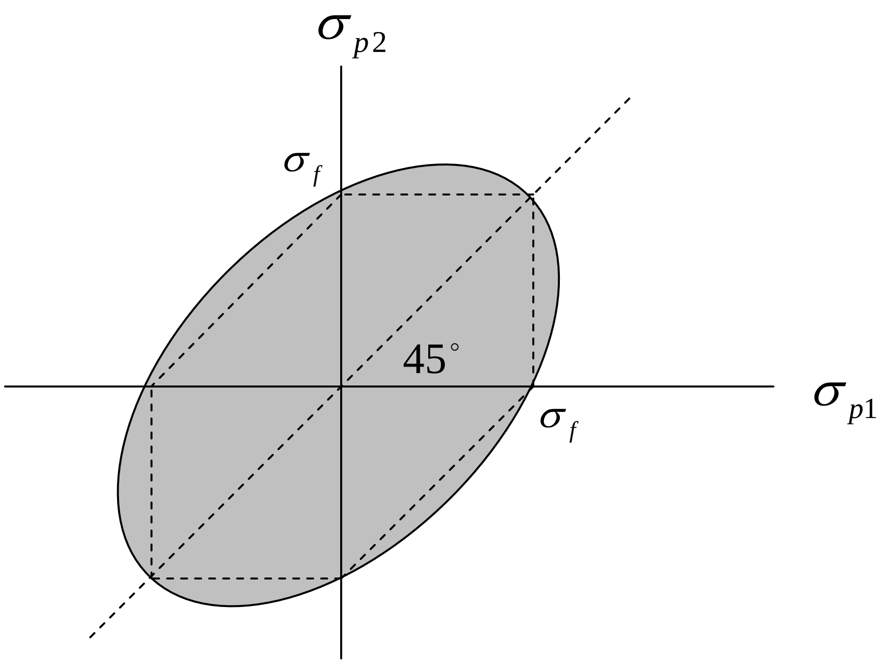
Maximum Distortional Strain Energy (von Mises) Theory

$u_d = u_f$ gives

$$\left(\sigma_{p1}-\sigma_{p2}\right)^2+\left(\sigma_{p1}-\sigma_{p3}\right)^2+\left(\sigma_{p2}-\sigma_{p3}\right)^2=2\sigma_f^2$$

For the biaxial (plane stress) case this reduces to

$$\sigma_{p1}^2-\sigma_{p1}\sigma_{p2}+\sigma_{p2}^2=\sigma_f^2 \quad \longleftarrow \quad \text{rotated ellipse}$$



Maximum Distortional Strain Energy (von Mises) Theory

Recall that the total shear stress on the octahedral plane is given by

$$\tau_{oct} = \frac{1}{3}\sqrt{\left(\sigma_{p1} - \sigma_{p2}\right)^2 + \left(\sigma_{p1} - \sigma_{p3}\right)^2 + \left(\sigma_{p2} - \sigma_{p3}\right)^2}$$

so that if we say that failure occurs when τ_{oct} equals the value of the octahedral stress at failure in a uniaxial tension test given by

$$\left(\tau_{oct}\right)_f = \frac{1}{3}\sqrt{2\sigma_f^2}$$

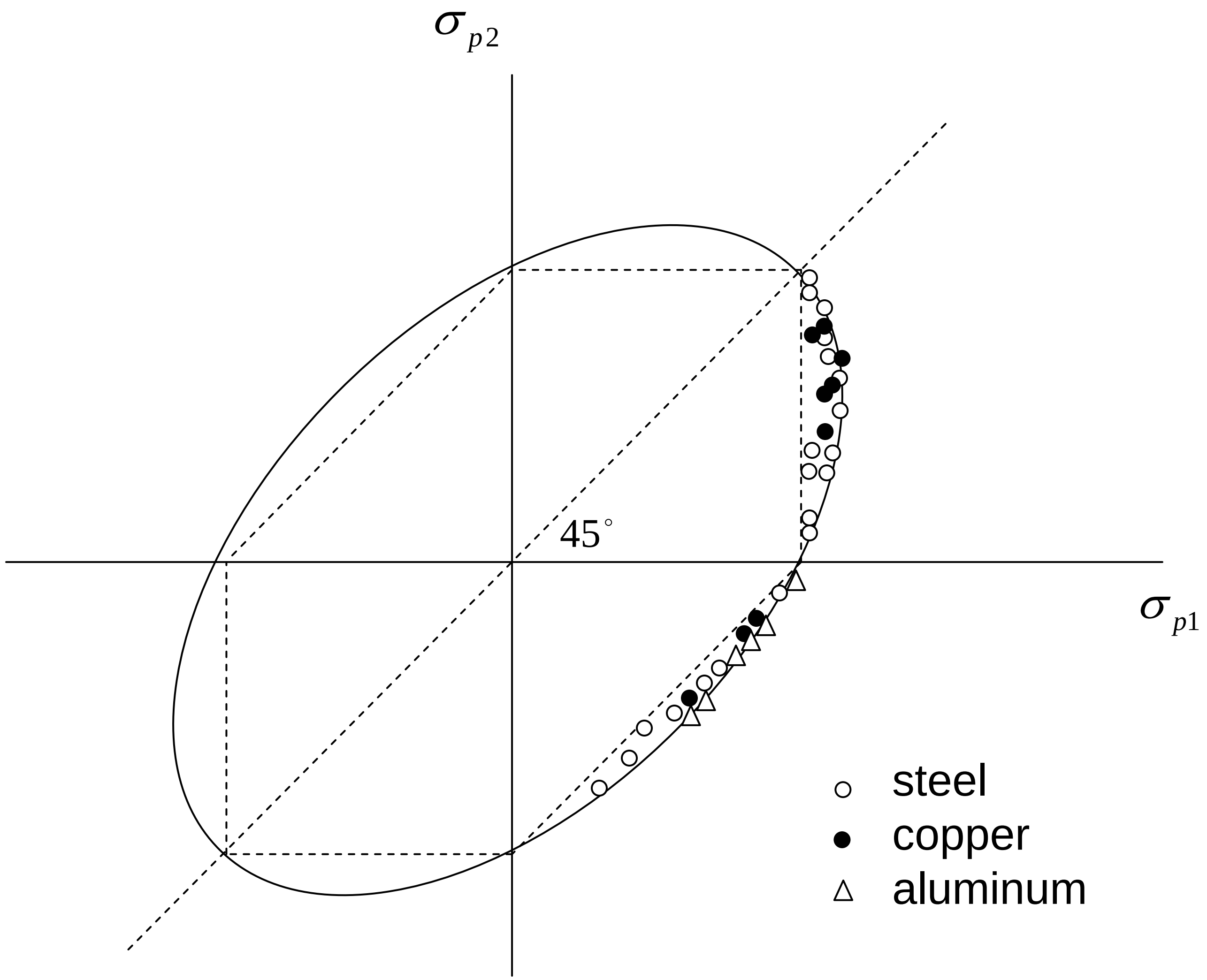
we get again

$$\left(\sigma_{p1} - \sigma_{p2}\right)^2 + \left(\sigma_{p1} - \sigma_{p3}\right)^2 + \left(\sigma_{p2} - \sigma_{p3}\right)^2 = 2\sigma_f^2$$

Maximum Distortional Strain Energy (von Mises) Theory

1. In formulating this failure theory we used generalized Hooke's law for an isotropic material so the theory given is only applicable to those materials but it can be generalized to anisotropic materials.

2. The von Mises theory is a little less conservative than the Tresca theory but in most cases there is little difference in their predictions of failure. Most experimental results tend to fall on or between these two theories.


σ_{p2}
45°
σ_{p1}
○ steel
● copper
△ aluminum

Tresca and von Mises Theories for 3-D stresses

Tresca

$$\tau_{\max} = \tau_f = \sigma_f / 2 = \max \begin{cases} \left|\sigma_{p1} - \sigma_{p2}\right| / 2 \\ \left|\sigma_{p1} - \sigma_{p3}\right| / 2 \\ \left|\sigma_{p2} - \sigma_{p3}\right| / 2 \end{cases}$$

von Mises

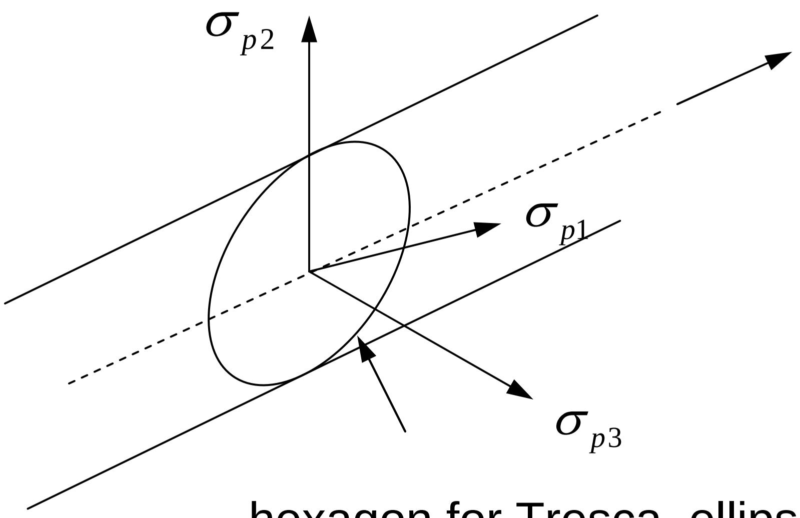
$$\left(\sigma_{p1} - \sigma_{p2}\right)^2 + \left(\sigma_{p1} - \sigma_{p3}\right)^2 + \left(\sigma_{p2} - \sigma_{p3}\right)^2 = 2\sigma_f^2$$

Both of these equations remain unchanged if we add equal principal stresses to all components, i.e.

$$\sigma'_{p1} = \sigma_{p1} + \sigma$$
$$\sigma'_{p2} = \sigma_{p2} + \sigma$$
$$\sigma'_{p3} = \sigma_{p3} + \sigma$$

since both theories are independent of adding or subtracting a hydrostatic component

This means that in 3-D the yield surfaces are cylinders whose sides are parallel to a line that makes equal angles with all three principal stress directions



$$\mathbf{n} = \frac{1}{\sqrt{3}}\mathbf{e}_{p1} + \frac{1}{\sqrt{3}}\mathbf{e}_{p2} + \frac{1}{\sqrt{3}}\mathbf{e}_{p3}$$

hexagon for Tresca, ellipse for von Mises