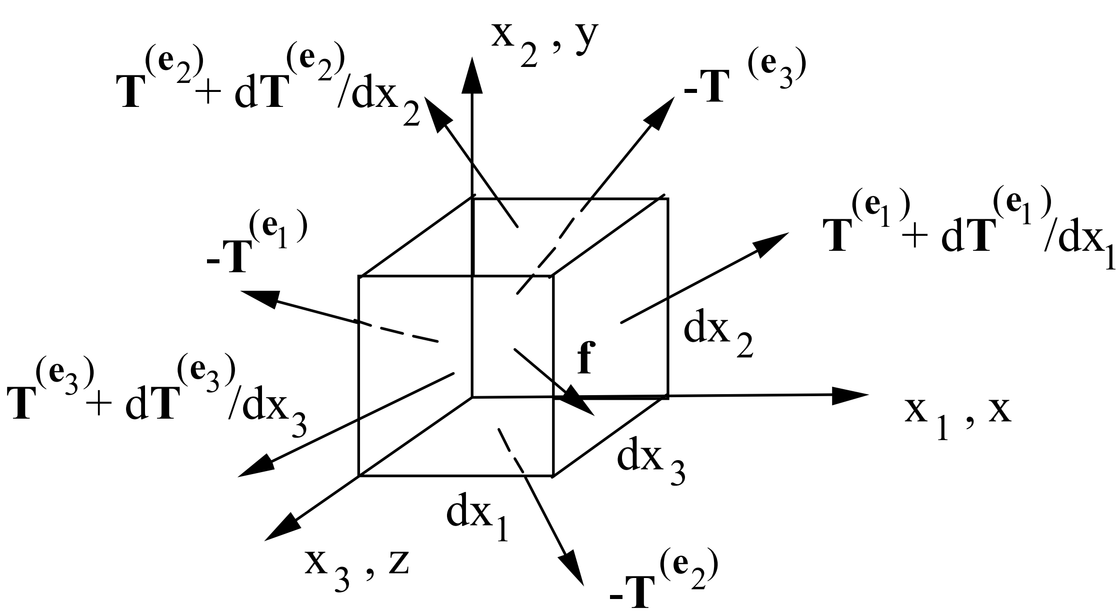
Equations of Equilibrium

Consider the small element of a body whose sides are cut out parallel to the x, y, z axes. Each of these sides has a traction vector acting on it, and since these tractions vary with position in a body, we must take into account those variations when we consider the tractions on each face. We also assume there is a body force **f** (force/unit volume) acting on the element.



Summing forces on the element we find

$$\left[\mathbf{T}^{(\mathbf{e}_1)} + \frac{\partial \mathbf{T}^{(\mathbf{e}_1)}}{\partial x_1} dx_1\right] dx_2 dx_3 - \mathbf{T}^{(\mathbf{e}_1)} dx_2 dx_3$$
$$+\left[\mathbf{T}^{(\mathbf{e}_2)} + \frac{\partial \mathbf{T}^{(\mathbf{e}_2)}}{\partial x_2} dx_2\right] dx_1 dx_3 - \mathbf{T}^{(\mathbf{e}_2)} dx_1 dx_3$$
$$+\left[\mathbf{T}^{(\mathbf{e}_3)} + \frac{\partial \mathbf{T}^{(\mathbf{e}_3)}}{\partial x_3} dx_3\right] dx_1 dx_2 - \mathbf{T}^{(\mathbf{e}_3)} dx_1 dx_2$$
$$+\mathbf{f}\, dx_1 dx_2 dx_3 = 0$$

which gives, when canceling terms and dividing by the volume $dx_1 dx_2 dx_3$ of the element

$$\frac{\partial \mathbf{T}^{(\mathbf{e}_1)}}{\partial x_1} + \frac{\partial \mathbf{T}^{(\mathbf{e}_2)}}{\partial x_2} + \frac{\partial \mathbf{T}^{(\mathbf{e}_3)}}{\partial x_3} + \mathbf{f} = 0$$

and which can be compactly written as

$$\sum_{i=1}^{3} \frac{\partial \mathbf{T}^{(\mathbf{e}_i)}}{\partial x_i} + \mathbf{f} = 0$$

If we express these traction vectors in terms of their component stresses with respect to the x, y, z axes and likewise express the body force in terms of its components, i.e.

$$\mathbf{T}^{(\mathbf{e}_i)} = \sum_{j=1}^{3} \sigma_{ij} \mathbf{e}_j$$
$$\mathbf{f} = \sum_{j=1}^{3} f_j \mathbf{e}_j$$

we find

$$\sum_{j=1}^{3} \left(\sum_{i=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_i} + f_j \right) \mathbf{e}_j = 0$$

and each component of this equation must vanish, leading to the three equations of local equilibrium for the stresses given by

$$\sum_{i=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} + f_j = 0 \qquad (j = 1, 2, 3)$$

which, when expanded out, become

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + f_x = 0$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + f_y = 0 \qquad (1)$$
$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z = 0$$

Equilibrium equations for plane stress and plane strain

For the special case of plane stress where $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$ and $f_z = 0$ these equations reduce to

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + f_x = 0$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y = 0 \qquad (2)$$

Similarly, for plane strain problems where $e_{zz} = e_{xz} = e_{yz} = 0$ from the stress strain relations it follows that for an isotropic elastic solid the shear stresses vanish. Since in plane strain problems we also assume $e_{xx} = e_{xx}(x,y), e_{xy} = e_{xy}(x,y), e_{yy} = e_{yy}(x,y)$, it follows also from the stress-strain equations that $\sigma_{zz} = \sigma_{zz}(x,y)$ so that with $f_z = 0$ the third equilibrium equation in Eq.(1) is again automatically satisfied and the other two equations just reduce to that of Eq. (2) for the plane stress case.