Equilibrium Equations in Cylindrical Coordinates

The equilibrium equations in Cartesian coordinates in terms of the traction vectors acting on the coordinate planes were found to be:

$$\sum_{i=1}^{3} \frac{\partial \mathbf{T}^{(\mathbf{e}_i)}}{\partial x_i} + \mathbf{f} = 0$$

where

$$\mathbf{T}^{(\mathbf{e}_i)} = \sum_{j=1}^{3} \sigma_{ij}\, \mathbf{e}_j \quad (i = 1,2,3)$$

We would like to be able to generalize these equations to other coordinate systems, with cylindrical coordinates in particular in mind. We can do this if we place these equations in a coordinate- invariant form. This can be done with the aid of the vector identity

$$\nabla \cdot (\phi \mathbf{A}) = \phi \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla \phi$$

To see this, first consider

$$\sum_{i=1}^{3} \frac{\partial B_i}{\partial x_i} = \nabla \cdot \mathbf{B} = \nabla \cdot \sum_{i=1}^{3} (B_i \mathbf{e}_i)$$

If we let $\phi = B_i$ and $\mathbf{A} = \mathbf{e}_i$ then we have

$$\begin{aligned} \sum_{i=1}^{3} \frac{\partial B_i}{\partial x_i} &= \sum_{i=1}^{3} B_i (\nabla \cdot \mathbf{e}_i) + (\mathbf{e}_i \cdot \nabla) B_i \\ &= \sum_{i=1}^{3} \left[(\nabla \cdot \mathbf{e}_i) + (\mathbf{e}_i \cdot \nabla) \right] B_i \end{aligned}$$

which is a coordinate-invariant form of the divergence. Now, consider

$$\begin{aligned} \sum_{i=1}^{3} \frac{\partial \mathbf{B}_i}{\partial x_i} &= \sum_{i-1}^{3} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{3} B_{ij} \mathbf{e}_j \right) \\ &= \sum_{i=1}^{3} \sum_{j=1}^{3} \left(\frac{\partial B_{ij}}{\partial x_i} \mathbf{e}_j + B_{ij} \frac{\partial \mathbf{e}_j}{\partial x_i} \right) \\ &= \sum_{i=1}^{3} \sum_{j=1}^{3} \left[\frac{\partial B_{ij}}{\partial x_i} \mathbf{e}_j + B_{ij} (\mathbf{e}_i \cdot \nabla) \mathbf{e}_j \right] \end{aligned}$$

But from our previous result on the divergence we can write

$$\sum_{i=1}^{3}\frac{\partial B_{ij}}{\partial x_i} = \sum_{i=1}^{3}\left[\left(\nabla\cdot\mathbf{e}_i\right)+\left(\mathbf{e}_i\cdot\nabla\right)\right]B_{ij}$$

so that we have

$$\begin{aligned}\sum_{i=1}^{3}\frac{\partial \mathbf{B}_i}{\partial x_i} &= \sum_{i=1}^{3}\sum_{j=1}^{3}\left[B_{ij}\mathbf{e}_j\left(\nabla\cdot\mathbf{e}_i\right)+\mathbf{e}_j\left(\mathbf{e}_i\cdot\nabla\right)B_{ij}+B_{ij}\left(\mathbf{e}_i\cdot\nabla\right)\mathbf{e}_j\right]\\ &= \sum_{i=1}^{3}\left[\mathbf{B}_i\left(\nabla\cdot\mathbf{e}_i\right)+\left(\mathbf{e}_i\cdot\nabla\right)\mathbf{B}_i\right]\\ &= \sum_{i=1}^{3}\left(\nabla\cdot\mathbf{e}_i+\mathbf{e}_i\cdot\nabla\right)\mathbf{B}_i\end{aligned}$$

which is the coordinate-invariant form we wanted. If we let $\mathbf{B}_i = \mathbf{T}^{(\mathbf{e}_i)}$ then the equilibrium equations in coordinate-invariant form are

$$\sum_{i=1}^{3}\left(\nabla\cdot\mathbf{e}_i+\mathbf{e}_i\cdot\nabla\right)\mathbf{T}^{(\mathbf{e}_i)}+\mathbf{f}=0$$

To use this result, let

$$\begin{aligned}\mathbf{e}_1 &\equiv \mathbf{e}_r = \cos\theta\,\mathbf{e}_x+\sin\theta\,\mathbf{e}_y\\ \mathbf{e}_2 &\equiv \mathbf{e}_\theta = -\sin\theta\,\mathbf{e}_x+\cos\theta\,\mathbf{e}_y\\ \mathbf{e}_3 &\equiv \mathbf{e}_z\end{aligned}$$

and

$$\begin{aligned}\mathbf{T}^{(\mathbf{e}_1)} &= \sigma_{rr}\mathbf{e}_r+\sigma_{r\theta}\mathbf{e}_\theta+\sigma_{rz}\mathbf{e}_z\\ \mathbf{T}^{(\mathbf{e}_2)} &= \sigma_{\theta r}\mathbf{e}_r+\sigma_{\theta\theta}\mathbf{e}_\theta+\sigma_{\theta z}\mathbf{e}_z\\ \mathbf{T}^{(\mathbf{e}_3)} &= \sigma_{zr}\mathbf{e}_r+\sigma_{z\theta}\mathbf{e}_\theta+\sigma_{zz}\mathbf{e}_z\\ \mathbf{f} &= f_r\mathbf{e}_r+f_\theta\mathbf{e}_\theta+f_r\mathbf{e}_r\end{aligned}$$

Using these results and the fact that in cylindrical coordinates

$$\nabla = \mathbf{e}_r\frac{\partial}{\partial r}+\mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\mathbf{e}_z\frac{\partial}{\partial z}$$

we find

$$\nabla \cdot \mathbf{e}_1 + \mathbf{e}_1 \cdot \nabla = \left(\frac{1}{r} + \frac{\partial}{\partial r}\right)$$
$$\nabla \cdot \mathbf{e}_2 + \mathbf{e}_2 \cdot \nabla = \frac{1}{r}\frac{\partial}{\partial \theta}$$
$$\nabla \cdot \mathbf{e}_3 + \mathbf{e}_3 \cdot \nabla = \frac{\partial}{\partial z}$$

so that the equilibrium equations are explicitly

$$\frac{\partial \mathbf{T}^{(\mathbf{e}_r)}}{\partial r} + \frac{\mathbf{T}^{(\mathbf{e}_r)}}{r} + \frac{1}{r}\frac{\partial \mathbf{T}^{(\mathbf{e}_\theta)}}{\partial \theta} + \frac{\partial \mathbf{T}^{(\mathbf{e}_z)}}{\partial z} + \mathbf{f} = 0$$

or, in terms of the stress components

$$\left(\frac{\partial \sigma_{rr}}{\partial r}\mathbf{e}_r + \frac{\partial \sigma_{r\theta}}{\partial r}\mathbf{e}_\theta + \frac{\partial \sigma_{rz}}{\partial r}\mathbf{e}_z\right) + \left(\frac{\sigma_{rr}}{r}\mathbf{e}_r + \frac{\sigma_{r\theta}}{r}\mathbf{e}_\theta + \frac{\sigma_{rz}}{r}\mathbf{e}_z\right)$$
$$+\frac{1}{r}\left(\frac{\partial \sigma_{\theta r}}{\partial \theta}\mathbf{e}_r + \frac{\partial \sigma_{\theta\theta}}{\partial \theta}\mathbf{e}_\theta + \frac{\partial \sigma_{\theta z}}{\partial \theta}\mathbf{e}_z + \sigma_{\theta r}\frac{\partial \mathbf{e}_r}{\partial \theta} + \sigma_{\theta\theta}\frac{\partial \mathbf{e}_\theta}{\partial \theta}\right)$$
$$+\left(\frac{\partial \sigma_{zr}}{\partial z}\mathbf{e}_r + \frac{\partial \sigma_{z\theta}}{\partial z}\mathbf{e}_\theta + \frac{\partial \sigma_{zz}}{\partial z}\mathbf{e}_z\right) + \left(f_r\mathbf{e}_r + f_\theta\mathbf{e}_\theta + f_z\mathbf{e}_z\right) = 0$$

which gives

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + f_r = 0$$
$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + f_\theta = 0$$
$$\frac{\partial \sigma_{zr}}{\partial r} + \frac{\sigma_{zr}}{r} + \frac{1}{r}\frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + f_z = 0$$

and, as with the Cartesian components of stress, we have, from moment equilibrium

$$\sigma_{r\theta} = \sigma_{\theta r}, \quad \sigma_{z\theta} = \sigma_{\theta z}, \quad \sigma_{zr} = \sigma_{rz}$$