STRAIN COMPATIBILITY EQUATIONS

Consider a body with displacements at points P_1 and P_2 given by $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively, as shown in Figure 1:

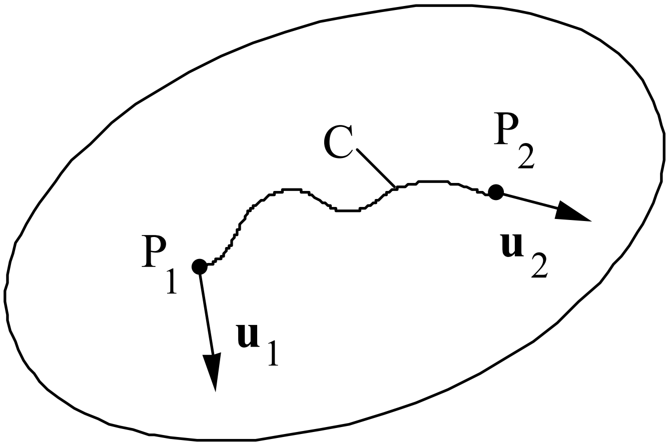

Figure 1

Let $\Delta = \mathbf{u}_2 - \mathbf{u}_1 =$ the relative displacement of P_2 with respect to P_1
Breaking that relative displacement into its components, we have

$$\Delta = \sum_{i=1}^{3} \Delta_i \mathbf{e}_i$$

Then those components can be written in terms of integrals of the local strains and rotations as:

$$\Delta_i = \int_{P_1}^{P_2} du_i = \sum_{j=1}^{3} \int_{P_1}^{P_2} \frac{\partial u_i}{\partial x_j} dx_j = \sum_{j=1}^{3} \int_{P_1}^{P_2} \left(\varepsilon_{ij} + \omega_{ij} \right) dx_j$$

But,

$$\omega_{ij} dx_j = d\left(\omega_{ij} x_j\right) - \sum_{k=1}^{3} x_j \frac{\partial \omega_{ij}}{\partial x_k} dx_k$$

so that

$$\Delta_i = \sum_{j=1}^{3} \omega_{ij} x_j \Big|_{P_1}^{P_2} + \sum_{k=1}^{3} \int_{P_1}^{P_2} \left(\varepsilon_{ik} - \sum_{j=1}^{3} x_j \frac{\partial \omega_{ij}}{\partial x_k} \right) dx_k$$

By differentiating the strain-displacement relationship it can be verified that these derivatives of the local rotation can also be written in terms of derivatives of the strains:

$$\frac{\partial \varepsilon_{ik}}{\partial x_j}-\frac{\partial \varepsilon_{jk}}{\partial x_i}=\frac{1}{2}\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right)=\frac{\partial \omega_{ij}}{\partial x_k}$$

giving

$$\Delta_i=\sum_{j=1}^{3}\omega_{ij}x_j\Big|_{P_1}^{P_2}+\sum_{k=1}^{3}\int_{P_1}^{P_2}\xi_{ik}\,dx_k$$

$$where \qquad \xi_{ik}=\varepsilon_{ik}-\sum_{j=1}^{3}x_j\left(\frac{\partial \varepsilon_{ik}}{\partial x_j}-\frac{\partial \varepsilon_{jk}}{\partial x_i}\right)$$

The first term in the above Δ_i expression only depends on P_1 and P_2 . The second term in Δ_i is independent of the path C between those two points (and hence only depends on P_1 and P_2) if, for a simply connected region (a region with no holes) we have

$$\frac{\partial \xi_{ik}}{\partial x_l}=\frac{\partial \xi_{il}}{\partial x_k} \tag{1}$$

(see Wylie, C.R., **Advanced Engineering Mathematics**, 4th Ed., McGraw-Hill, p. 684). Thus, if Eq. (1) is satisfied everywhere in a simply connected region the displacement will have a single value everywhere in that region (the displacement is *single valued* but not unique since we can always add a rigid body displacement that does not change the strains). The derivatives contained in Eq. (1) can be written as

$$\begin{aligned}\frac{\partial \xi_{ik}}{\partial x_l}&=\frac{\partial \varepsilon_{ik}}{\partial x_l}-\left(\frac{\partial \varepsilon_{ik}}{\partial x_l}-\frac{\partial \varepsilon_{lk}}{\partial x_i}\right)-\sum_{j=1}^{3}x_j\left(\frac{\partial^2 \varepsilon_{ik}}{\partial x_j\partial x_l}-\frac{\partial^2 \varepsilon_{jk}}{\partial x_i\partial x_l}\right)\\ \frac{\partial \xi_{il}}{\partial x_k}&=\frac{\partial \varepsilon_{il}}{\partial x_k}-\left(\frac{\partial \varepsilon_{il}}{\partial x_k}-\frac{\partial \varepsilon_{kl}}{\partial x_i}\right)-\sum_{j=1}^{3}x_j\left(\frac{\partial^2 \varepsilon_{il}}{\partial x_j\partial x_k}-\frac{\partial^2 \varepsilon_{jl}}{\partial x_i\partial x_k}\right)\end{aligned} \tag{2}$$

However, the first two terms in the above equations cancel and the third terms in the above equations also cancel when placed back into Eq. (1), reducing Eq. (1) to:

$$\sum_{j=1}^{3} x_j \left(\frac{\partial^2 \varepsilon_{ik}}{\partial x_j \partial x_l} + \frac{\partial^2 \varepsilon_{jl}}{\partial x_i \partial x_k} - \frac{\partial^2 \varepsilon_{jk}}{\partial x_i \partial x_l} - \frac{\partial^2 \varepsilon_{il}}{\partial x_j \partial x_k} \right) = 0$$

But, since the x_j are independent, the quantity in the brackets must vanish, and

$$\frac{\partial^2 \varepsilon_{ik}}{\partial x_j \partial x_l} + \frac{\partial^2 \varepsilon_{jl}}{\partial x_i \partial x_k} - \frac{\partial^2 \varepsilon_{jk}}{\partial x_i \partial x_l} - \frac{\partial^2 \varepsilon_{il}}{\partial x_j \partial x_k} = 0 \tag{3}$$

We will write Eq.(3) symbolically as

$$R_{ijkl} = 0 \tag{4}$$

Since *i, j, k,* and *l* can all have values ranging from 1 to 3, Eq. (4) looks like a total of 81 equations. However, because of the following symmetries and anti-symmetries:

$$\begin{aligned} R_{ijkl} &= R_{klij} \\ R_{ijkl} &= -R_{jikl} = -R_{ijlk} \end{aligned}$$

it turns out that there are really only six distinct terms in Eq. (4) given by the compatibility conditions:

$$\begin{aligned} S_{11} &\equiv R_{2323} = 0 \\ S_{22} &\equiv R_{3131} = 0 \\ S_{33} &\equiv R_{1212} = 0 \\ S_{21} &= S_{12} \equiv R_{2331} = 0 \\ S_{23} &= S_{32} \equiv R_{3112} = 0 \\ S_{13} &= S_{31} \equiv R_{1223} = 0 \end{aligned} \tag{5}$$

which we have written in terms of the components of a symmetric *S* matrix:

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

All six of the compatibility equations listed in Eq. (5), however, are not independent since the components of the *S* matrix can be shown to satisfy the three equations:

$$\sum_{j=1}^{3}\frac{\partial S_{ij}}{\partial x_j}=0 \quad (i=1,2,3)$$

so that these six compatibility equations really represent only three independent conditions that the strains must satisfy. Explicitly, these six equations are

$$2\frac{\partial^2\varepsilon_{23}}{\partial x_2\partial x_3}-\frac{\partial^2\varepsilon_{22}}{\partial x_3^2}-\frac{\partial^2\varepsilon_{33}}{\partial x_2^2}=0$$

$$2\frac{\partial^2\varepsilon_{31}}{\partial x_3\partial x_1}-\frac{\partial^2\varepsilon_{33}}{\partial x_1^2}-\frac{\partial^2\varepsilon_{11}}{\partial x_3^2}=0$$

$$2\frac{\partial^2\varepsilon_{12}}{\partial x_1\partial x_2}-\frac{\partial^2\varepsilon_{11}}{\partial x_2^2}-\frac{\partial^2\varepsilon_{22}}{\partial x_1^2}=0$$

$$\frac{\partial^2\varepsilon_{33}}{\partial x_1\partial x_2}+\frac{\partial^2\varepsilon_{12}}{\partial x_3^2}-\frac{\partial^2\varepsilon_{23}}{\partial x_3\partial x_1}-\frac{\partial^2\varepsilon_{31}}{\partial x_3\partial x_2}=0$$

$$\frac{\partial^2\varepsilon_{11}}{\partial x_2\partial x_3}+\frac{\partial^2\varepsilon_{23}}{\partial x_1^2}-\frac{\partial^2\varepsilon_{31}}{\partial x_1\partial x_2}-\frac{\partial^2\varepsilon_{12}}{\partial x_1\partial x_3}=0$$

$$\frac{\partial^2\varepsilon_{22}}{\partial x_3\partial x_1}+\frac{\partial^2\varepsilon_{31}}{\partial x_2^2}-\frac{\partial^2\varepsilon_{12}}{\partial x_2\partial x_3}-\frac{\partial^2\varepsilon_{23}}{\partial x_2\partial x_1}=0$$

Because these six equations are not independent, it is difficult in general 3-D problems to use these compatibility equations directly. However, for plane strain problems, we have only two non-zero displacements since plane strain conditions require

$$\begin{aligned} u_1 &= u_1(x_1,x_2)\\ u_2 &= u_2(x_1,x_2)\\ u_3 &= 0 \end{aligned}$$

which implies that

$$\begin{aligned} \varepsilon_{33} &= \varepsilon_{13}=\varepsilon_{23}=0\\ \varepsilon_{11} &= \varepsilon_{11}(x_1,x_2)\\ \varepsilon_{22} &= \varepsilon_{22}(x_1,x_2)\\ \varepsilon_{12} &= \varepsilon_{12}(x_1,x_2) \end{aligned}$$

so there is only one compatibility equation that is not identically zero given by

$$2\frac{\partial^2\varepsilon_{12}}{\partial x_1\partial x_2}-\frac{\partial^2\varepsilon_{11}}{\partial x_2^2}-\frac{\partial^2\varepsilon_{22}}{\partial x_1^2}=0$$

For plane stress problems, on the other hand, we have the conditions

$$\begin{aligned}
&\sigma_{13} = \sigma_{23} = \sigma_{33} = 0\\
&\sigma_{11} = \sigma_{11}(x_1, x_2)\\
&\sigma_{22} = \sigma_{22}(x_1, x_2)\\
&\sigma_{12} = \sigma_{12}(x_1, x_2)
\end{aligned}$$

which implies that

$$\begin{aligned}
&\varepsilon_{13} = \varepsilon_{23} = 0 \quad \textit{but} \quad \varepsilon_{33} \neq 0\\
&\varepsilon_{11} = \varepsilon_{11}(x_1, x_2)\\
&\varepsilon_{22} = \varepsilon_{22}(x_1, x_2)\\
&\varepsilon_{12} = \varepsilon_{12}(x_1, x_2)
\end{aligned}$$

so that the compatibility equations reduce to

$$\begin{aligned}
&\frac{\partial^2 \varepsilon_{33}}{\partial x_2^2} = 0\\
&\frac{\partial^2 \varepsilon_{33}}{\partial x_1^2} = 0\\
&\frac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2} = 0\\
&2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} - \frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} - \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = 0
\end{aligned}$$

In general, the first three of the above conditions cannot be satisfied exactly, so that plane stress conditions can only be approximately true in a body. For thin bodies, however, this approximation can be usually justified. Note that the compatibility equation for the in-plane strains in the case of plane stress is identical to the compatibility equation for plane strain.

For bodies with holes these compatibility equations are not sufficient to guarantee that the strains can be obtained from a single-valued displacement field. In fact, in bodies with holes there may be some cases where we want to have displacements that are not single valued such as the split ring shown in Fig.2:

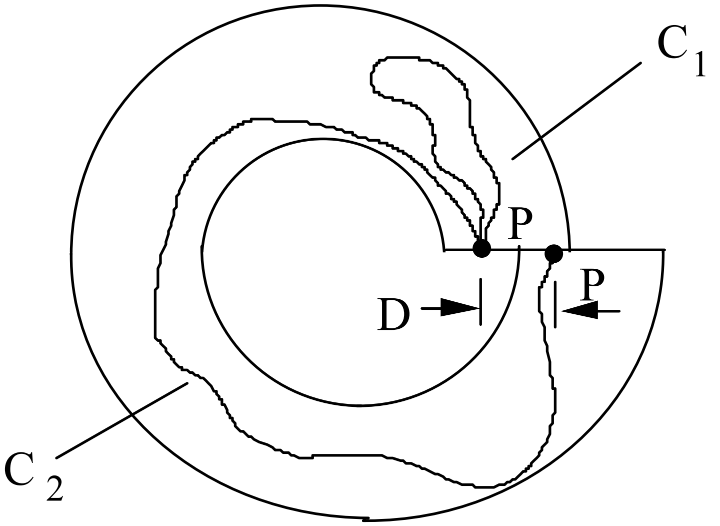

Fig.2

where the point P on one side of the split is fixed (displacement = 0) and the x-displacement of the same point P on the other side of the split = D. In this case, we see that

$$\int_{C_1} du_j = 0$$

$$\int_{C_2} du_j = \begin{cases} D & j=1 \\ 0 & j=2,3 \end{cases}$$

To ensure that a multiply connected body like the ring shown above cannot split in this fashion, we must supplement the compatibility equations by additional conditions. For a body with *m* holes as shown in Fig. 3 if, in addition to the compatibility equations, we require the *m* subsidiary conditions,

$$\int_{C_i} du_j = 0 \quad (i = 1, 2, \ldots, m)$$

where the integrals are taken around each hole, then the displacements will be single-valued.

Note that in solving any problem, if we end up obtaining directly a displacement field that represents the solution to the desired problem, then compatibility is not an issue since we can always generate a set of strains that are compatible with those displacements simply by taking the appropriate displacement derivatives. When we directly solve a problem only for the strains (or corresponding stresses), however, it is not automatically guaranteed that a well-behaved, single-valued displacement field can be obtained through integrating those strains. The compatibility equations (and any subsidiary conditions needed for multiply connected bodies) provide the guarantee that a well-behaved, single-valued displacement field can be obtained from the given strain field.

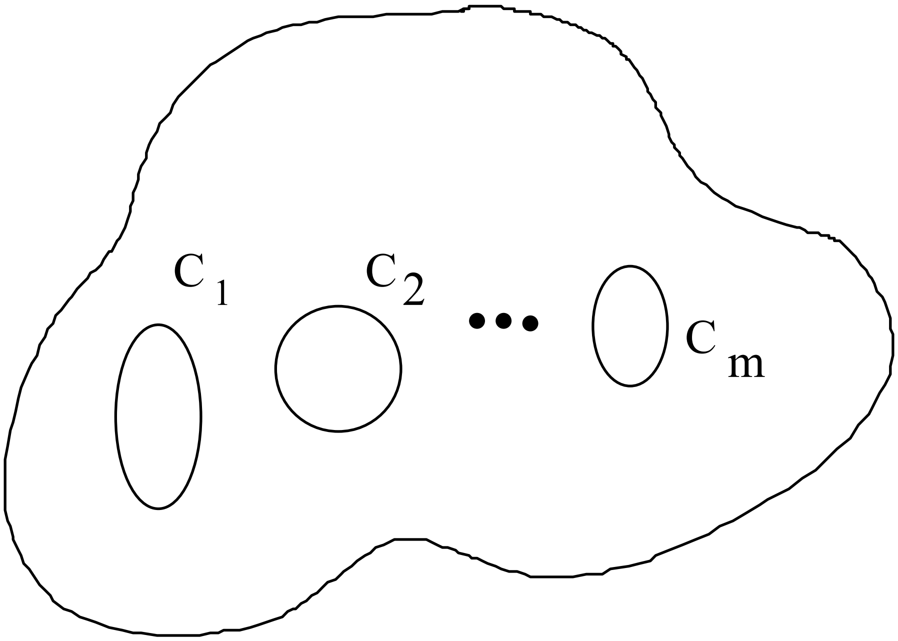

Fig. 3