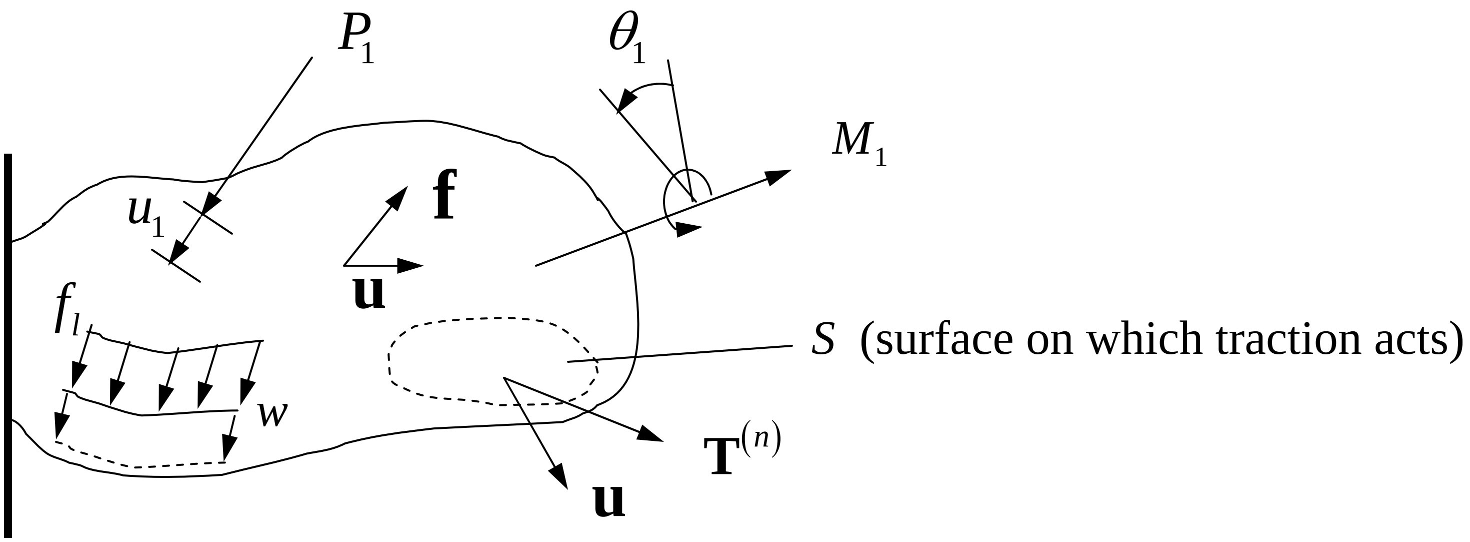
Complimentary Virtual Work   3-D Problems



$$\delta W_v^c = \int_S \delta \mathbf{T}^{(n)} \cdot \mathbf{u}\, dS + \int_V \delta \mathbf{f} \cdot \mathbf{u}\, dV + \int_l \delta f_l w ds + \underline{\sum_i \delta P_i u_i + \sum_i \delta M_i \theta_i}$$

for multiple loads and moments

As in the 1-D case can show

$$\delta W_v^c = \int_V \sum_{i=1}^{3} \sum_{j=1}^{3} e_{ij} \delta \sigma_{ij}\, dV = \delta U^c$$

Principle of Complimentary Virtual Work

If the principle of complementary virtual work is satisfied for all variations of the stresses, where those variations satisfy local equilibrium, then the compatibility equations will also be satisfied

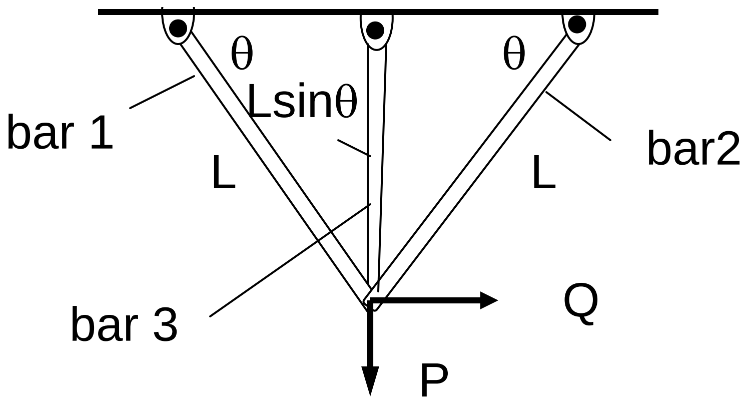
$$\delta W_v^c = \delta U^c$$

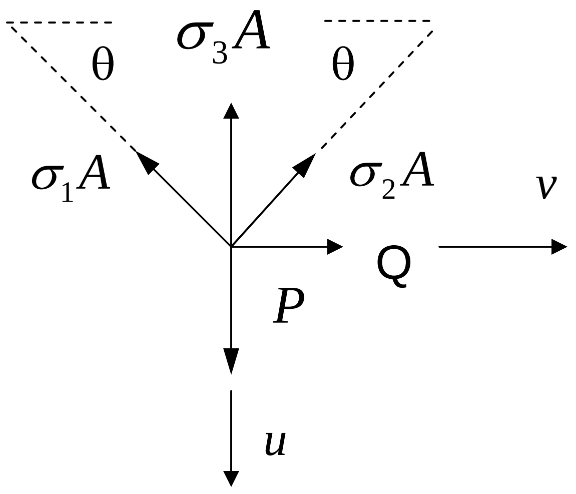
for all possible stresses
satisfying local equilibrium

⇒

compatibility of
the stresses
(and strains)

Determine the stresses in the three elastic bars by the principle of complimentary virtual work. All bars have the same E, A





Equilibrium Eqs:

$$\sigma_2 A\sin\theta - \sigma_1 A\sin\theta + Q = 0$$

$$\sigma_1 A\cos\theta + \sigma_2 A\cos\theta + \sigma_3 A - P = 0$$

Complimentary Strain Energy for a linear elastic bar of length l

$$U^c = U = \frac{Al\sigma^2}{2E}$$

Variations must satisfy

$$\delta\sigma_2 A\sin\theta - \delta\sigma_1 A\sin\theta + \delta Q = 0$$
$$\delta\sigma_1 A\cos\theta + \delta\sigma_2 A\cos\theta + \delta\sigma_3 A - \delta P = 0$$

and for a single bar $$\delta U^c = \frac{Al\sigma\delta\sigma}{E} = A\Delta\delta\sigma$$

so from $$\delta W_v^c = \delta U^c = u\delta P + v\delta Q$$

$$A\Delta_1\delta\sigma + A\Delta_2\delta\sigma + A\Delta_3\delta\sigma_3 = \overbrace{\left(\delta\sigma_1 uA\cos\theta + \delta\sigma_2 uA\cos\theta + \delta\sigma_3 u\right)}^{u\delta P}$$
$$\underbrace{-\left(v\delta\sigma_2 A\sin\theta - v\delta\sigma_1 A\sin\theta\right)}_{v\delta Q}$$

From the principle of complimentary virtual work we found

$$A\Delta_1\delta\sigma_1 + A\Delta_2\delta\sigma_2 + A\Delta_3\delta\sigma_3 = \delta\sigma_1 uA\cos\theta + \delta\sigma_2 uA\cos\theta + \delta\sigma_3 Au$$
$$-v\delta\sigma_2 A\sin\theta + v\delta\sigma_1 A\sin\theta$$

satisfying this for all $\delta\sigma_1, \delta\sigma_2, \delta\sigma_3$

gives

$$\Delta_1 = u\cos\theta + v\sin\theta$$
$$\Delta_2 = u\cos\theta - v\sin\theta$$
$$\Delta_3 = u$$

so these are the compatible bar elongations (compatible with the two displacements u, v, at the applied loads) . We see

$$\Delta_1 + \Delta_2 = 2\Delta_3\cos\theta$$

$$\Delta_1 + \Delta_2 = 2\Delta_3 \cos\theta$$

This equation is the compatibility equation for our problem. It shows that the three elongations in the bar must be related in order that the three bars all give consistent end displacements (u, v). We could also write this equation in terms of compatibility of the strains in the bars

$$e_1 L + e_2 L = 2 e_3 L \sin\theta \cos\theta$$

Using Hooke's law, in terms of the stresses we find

$$\frac{\sigma_1 L}{E} + \frac{\sigma_2 L}{E} = \frac{2\sigma_3 L \sin\theta}{E} \cos\theta$$

Thus, we have solved the problem by the method of complimentary virtual work since from equilibrium

$$\sigma_2 A \sin\theta - \sigma_1 A \sin\theta + Q = 0$$

$$\sigma_1 A \cos\theta + \sigma_2 A \cos\theta + \sigma_3 A - P = 0$$

and from compatibility

$$\frac{\sigma_1 L}{E} + \frac{\sigma_2 L}{E} = \frac{2\sigma_3 L \sin\theta}{E}\cos\theta$$

we have three equations in three unknowns