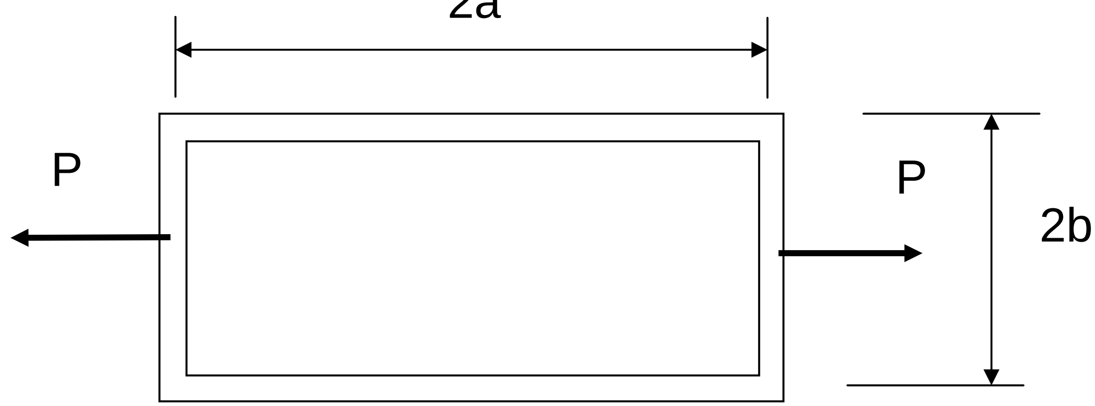
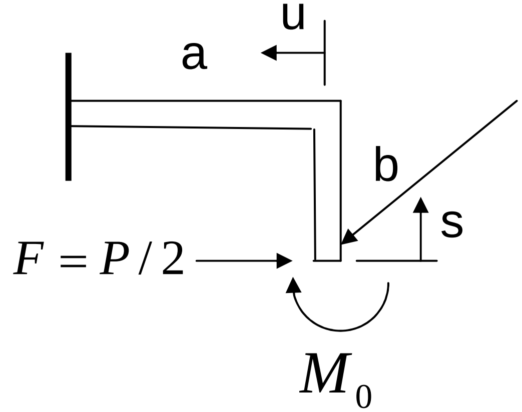
Use of the Principle of Least Work and Castigliano's Second Theorem



Determine the elongation of the frame between the points where the two loads are applied

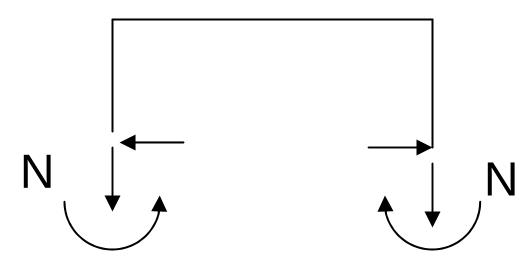
For simplicity consider one quarter of the frame



there is no shear stress at the cut because of symmetry and no normal force, N,

because of symmetry and

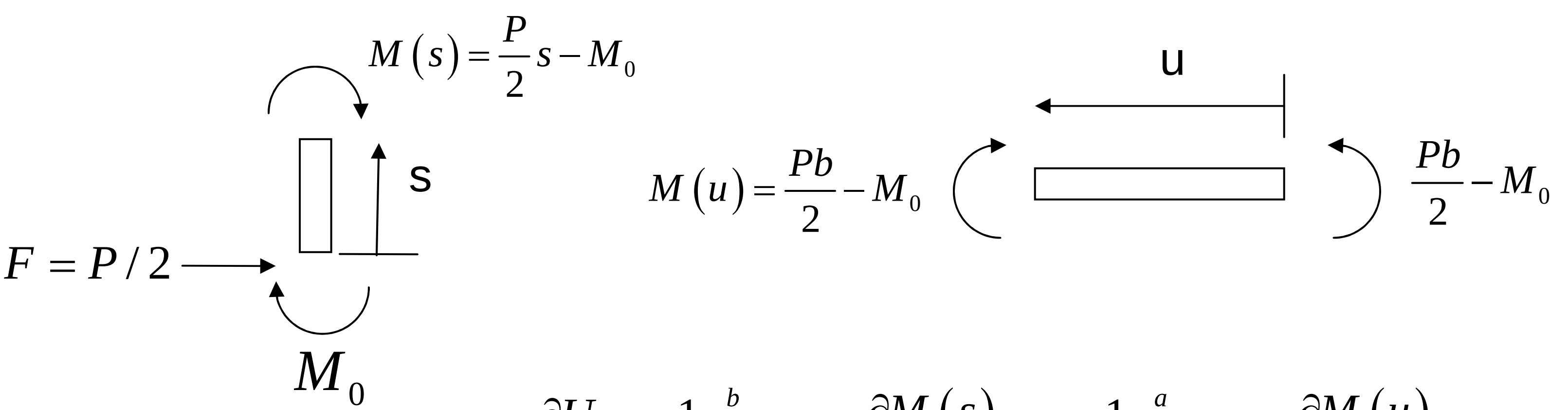
$$\uparrow \sum F_y = 0$$



We don't know the value of the internal moment M_0, so this problem is statically indeterminant. From the Principle of Least Work

$$\delta U = \frac{\partial U}{\partial M_0}\delta M_0 = 0 \qquad \text{for arbitrary } \delta M_0$$

Here, we keep only the strain energy due to bending $U = \frac{1}{2EI}\int M^2(x)\,dx$



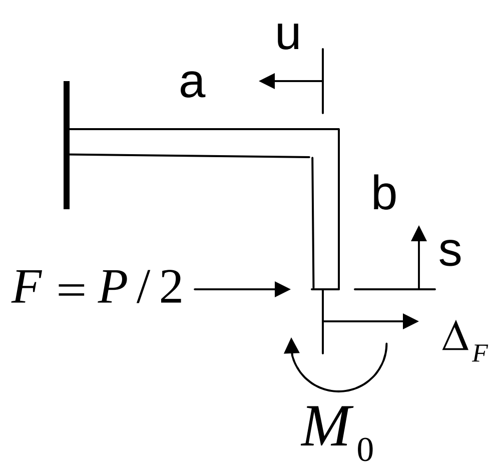
Thus, $$\frac{\partial U}{\partial M_0} = \frac{1}{EI}\int_0^b M(s)\frac{\partial M(s)}{\partial M_0}ds + \frac{1}{EI}\int_0^a M(u)\frac{\partial M(u)}{\partial M_0}du = 0$$

which gives $$\frac{1}{EI}\int_0^b\left(\frac{Ps}{2} - M_0\right)(-1)\,ds + \frac{1}{EI}\int_0^a\left(\frac{Pb}{2} - M_0\right)(-1)\,du = 0$$

$$\Rightarrow \quad \frac{-Pb^2}{4} + M_0 b - \frac{Pba}{2} + M_0 a = 0$$

Thus, we obtain $$M_0 = \frac{Pb\left(2a+b\right)}{4\left(a+b\right)}$$

Now that we have the moment we can find the displacement Δ_F



Note that $$U = U\left(F, M_0\left(F\right)\right)$$

But we have $$\Delta_F = \left.\frac{\partial U}{\partial F}\right|_{M_0} + \left.\frac{\partial U}{\partial M_0}\right|_F \frac{\partial M_0}{\partial F}$$

Thus, even though $M_0 = M_0\left(F\right)$ we can treat M_0 as if it was a constant when differentiating the strain energy

$$U = \frac{1}{2EI}\int_0^b \left(Fs - M_0\right)^2 ds + \frac{1}{2EI}\int_0^a \left(Fb - M_0\right)^2 du$$

$$\Delta_F = \frac{\partial U}{\partial F} = \frac{1}{EI}\int_0^b \left(Fs - M_0\right) s ds + \frac{1}{EI}\int_0^a \left(Fb - M_0\right) b du$$

$$= \frac{1}{EI}\left(\frac{Fb^3}{3} - \frac{M_0 b^2}{2} + Fab^2 - M_0 ab\right)$$

Placing in the expressions for F and M_0 in terms of P

$$\Delta_F = \frac{Pb^3\left(4a + b\right)}{24EI\left(a + b\right)}$$

We need to double this result since this is only the elongation of one side, giving

$$\Delta_P = 2\Delta_F = \frac{Pb^3\left(4a + b\right)}{12EI\left(a + b\right)}$$