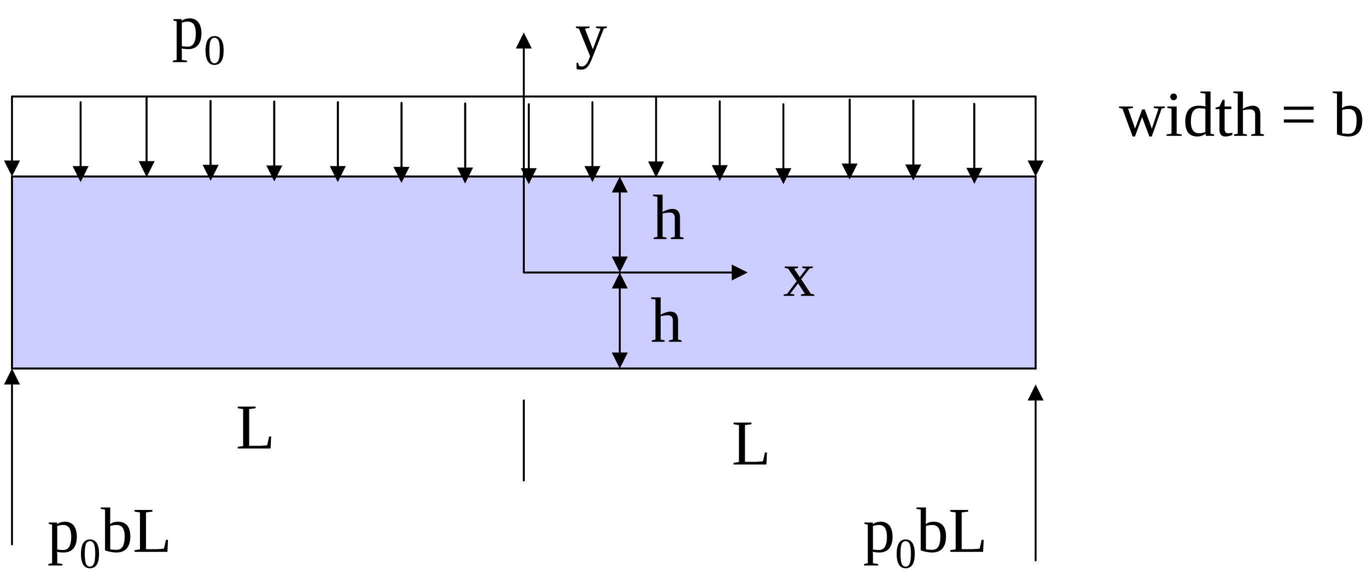


$$\phi = -\frac{p_0 x^2}{4} + a_2 x^2 + b_3 x^2 y + d_3 y^3 + b_5 x^4 y + d_5 x^2 y^3 + f_5 y^5$$

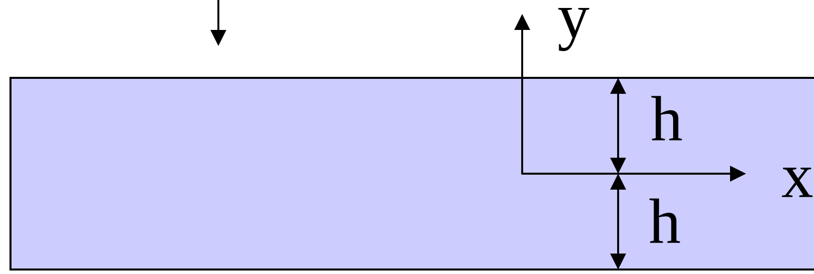
six unknown constants

$$\nabla^4 \phi = 0 \quad \Rightarrow \quad 24 b_5 + 24 d_5 + 120 f_5 = 0 \qquad (1)$$

Boundary Conditions

$$\sigma_{yy}(x, h) = -p_0 \qquad \text{(I)}$$

$$\sigma_{xy}(x, h) = 0 \qquad \text{(II)}$$



$$\sigma_{yy}(x, -h) = 0 \qquad \text{(III)}$$

$$\sigma_{xy}(x, -h) = 0 \qquad \text{(IV)}$$

The stresses

$$\sigma_{xx}(x,y) = 6d_3 y + 6d_5 x^2 y + 20 f_5 y^3$$

$$\sigma_{yy}(x,y) = -p_0/2 + 2a_2 + 2b_3 y + 12b_5 x^2 y + 2d_5 y^3$$

$$\sigma_{xy}(x,y) = -\left(2b_3 x + 4b_5 x^3 + 6d_5 x y^2\right)$$

Now, apply the boundary conditions

(I) : $\sigma_{yy}(x,h) = -p_0$

$$\Rightarrow \quad 2a_2 + 2b_3 h + 12b_5 x^2 h + 2d_5 h = -p_0/2$$

$$\Rightarrow \left| \begin{array}{ll} b_5 = 0 & (2) \\ 2a_2 + 2b_3 h + 2d_5 h = -p_0/2 & (3) \end{array} \right.$$

$$(\text{II}) : \sigma_{xy}\left(x,h\right)=0 \quad \Rightarrow \quad b_3 = -3d_5h^2 \qquad (4)$$

$$(\text{III}) : \sigma_{yy}\left(x,-h\right)=0$$

$$\Rightarrow \quad 2a_2 - 2b_3h - 2d_5h^3 = -p_0 / 2 \qquad (5)$$

$$(\text{IV}): \sigma_{xy}\left(x,-h\right)=0 \quad \Rightarrow \quad b_3 = -3d_5h^2 \qquad (4) \text{ again}$$

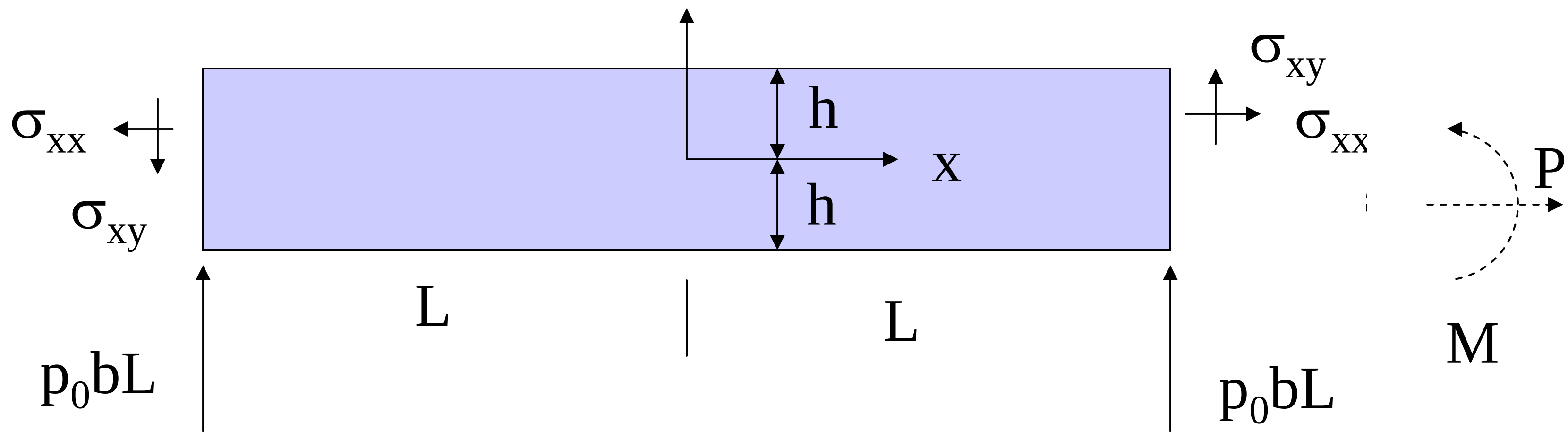
From equations (1) – (5):

$$a_2 = 0 \qquad d_5 = \frac{p_0}{8h^3}$$

$$b_3 = \frac{-3p_0}{8h} \qquad f_5 = \frac{-p_0}{40h^3}$$

$$b_5 = 0$$

so only d_3 remains to be found



Saint-Venant boundary conditions

x = -L

(d) $$\int_{-h}^{+h} \sigma_{xy} b\,dy = -p_0 bL$$

(e) $$\int_{-h}^{+h} \sigma_{xx} b\,dy = 0$$

(f) $$\int_{-h}^{+h} \sigma_{xx} y b\,dy = 0$$

x = L

(a) $$\int_{-h}^{+h} \sigma_{xy} b\,dy = p_0 bL$$

(b) $$\int_{-h}^{+h} \sigma_{xx} b\,dy = 0$$ (P = 0)

(c) $$\int_{-h}^{+h} \sigma_{xx} y b\,dy = 0$$ (M = 0)

(a) : $$\int_{-h}^{+h}\sigma_{xy}b\,dy = p_0 bL \quad\Rightarrow\quad -\left(4b_3h+4d_5h^3\right)bL = p_0 bL$$

already satisfied

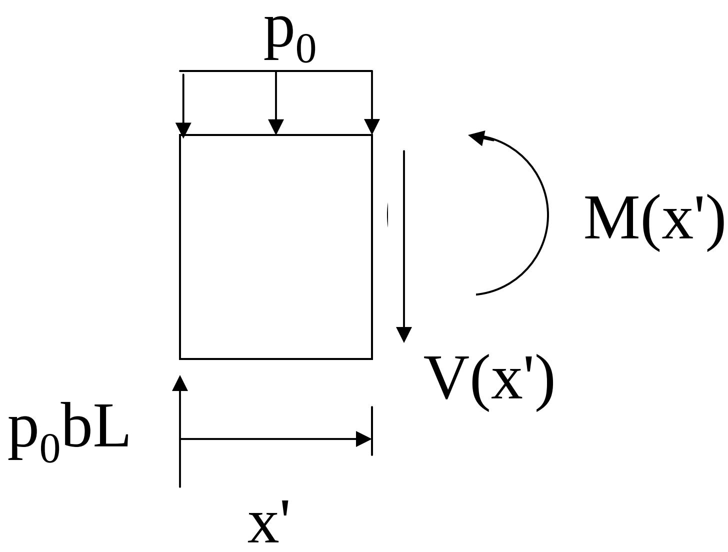
(b) : $$\int_{-h}^{+h}\sigma_{xx}b\,dy = 0$$

satisfied automatically
(σ_{xx} is odd in y)

(c) : $$\int_{-h}^{+h}\sigma_{xx}yb\,dy = 0 \quad\Rightarrow\quad 4d_3h^3+2d_5L^2h^3+8f_5h^5=0$$

$$\Rightarrow\quad d_3=\frac{p_0}{8h}\left(\frac{2}{5}-\frac{L^2}{h^2}\right)$$

(d), (e), (f) are all also satisfied with these same constants



$$\sigma_{xx} = -\frac{M(x')\,y}{I} + p_0\left(\frac{6y}{20h} - \frac{y^3}{2h^3}\right)$$

$$\sigma_{xy} = -\frac{V(x')\,Q(y)}{Ib} \qquad \uparrow +$$

$$\sigma_{yy} = -p_0\left(\frac{1}{2} + \frac{3y}{4h} - \frac{y^3}{4h^3}\right)$$

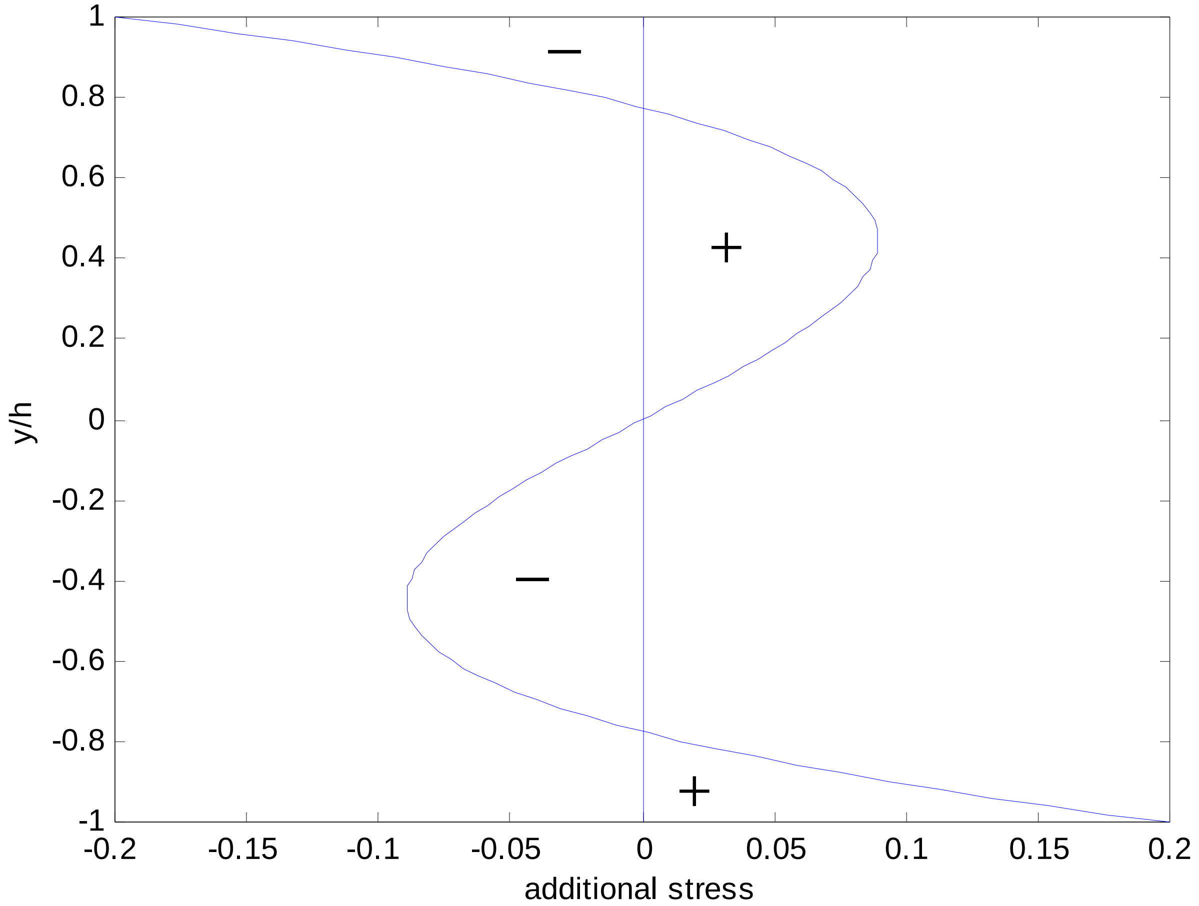
$$\sigma_{xx}\Big|_{\max}^{strength} = \frac{3p_0L^2}{4h^2}$$ At x = L (center of beam)

$$\sigma_{xx}\Big|_{\max}^{add} = \frac{p_0}{5} \qquad \sigma_{yy}\Big|_{\max}^{add} = p_0$$

$$\Rightarrow \quad \frac{\sigma_{xx}\Big|_{\max}^{add}}{\sigma_{xx}\Big|_{\max}^{strength}} = \frac{4}{15}\frac{h^2}{L^2} \qquad \frac{\sigma_{yy}\Big|_{\max}^{add}}{\sigma_{xx}\Big|_{\max}^{strength}} = \frac{4}{3}\frac{h^2}{L^2}$$

The additional flexure stress
(self equilibrated: P = M =0)

$$\frac{\sigma_{xx}}{p_0} = \left(\frac{6y}{20h} - \frac{y^3}{2h^3} \right)$$



The additional stress

$$\frac{\sigma_{yy}}{p_0} = -\left(\frac{1}{2} + \frac{3y}{4h} - \frac{y^3}{4h^3}\right)$$