Airy Stress Function (Plane Stress Problems)

Equilibrium

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + f_x = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y = 0$$

Let

$$f_x = -\frac{\partial V}{\partial x} \qquad \sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} + V$$

$$f_y = -\frac{\partial V}{\partial y} \qquad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} + V$$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

V, potential function

ϕ , Airy stress function

Then equilibrium conditions are automatically satisfied

Strains from stress-strain relations

$$e_{xx} = \frac{1}{E}\left[\frac{\partial^2 \phi}{\partial y^2} - \nu \frac{\partial^2 \phi}{\partial x^2} + (1-\nu)V\right]$$

$$e_{yy} = \frac{1}{E}\left[\frac{\partial^2 \phi}{\partial x^2} - \nu \frac{\partial^2 \phi}{\partial y^2} + (1-\nu)V\right]$$

$$\gamma_{xy} = -\frac{2(1+\nu)}{E}\frac{\partial^2 \phi}{\partial x \partial y}$$

Placing these strains into the compatibility equations

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

gives

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x \partial y} + \frac{\partial^4 \phi}{\partial y^4} = -(1-\nu)\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right)$$

or

$$\nabla^4 \phi = -(1-\nu)\nabla^2 V$$

biharmonic operator

Laplacian operator

compatibility equation in terms of the Airy stress function

$$\nabla^4 \phi = -(1-\nu)\nabla^2 V$$

compatibility in terms of stresses

$$\nabla^2 \left(\sigma_{xx} + \sigma_{yy} \right) = (1+\nu)\nabla^2 V$$

For no body forces

$$\nabla^4 \phi = 0$$

or $$\nabla^2 \left(\sigma_{xx} + \sigma_{yy} \right) = 0$$

Polynomial Solutions of the Homogeneous Biharmonic Equation

$$\nabla^4 \phi = 0$$

$$\phi = \sum_{n=1}^{N} \sum_{m=1}^{M} A_{mn} x^n y^m$$

Polynomials up to third order (cubics) satisfy the biharmonic equation without any restrictions on the A's. Fourth order and higher polynomials require that those coefficients be related to satisfy the biharmonic equation.