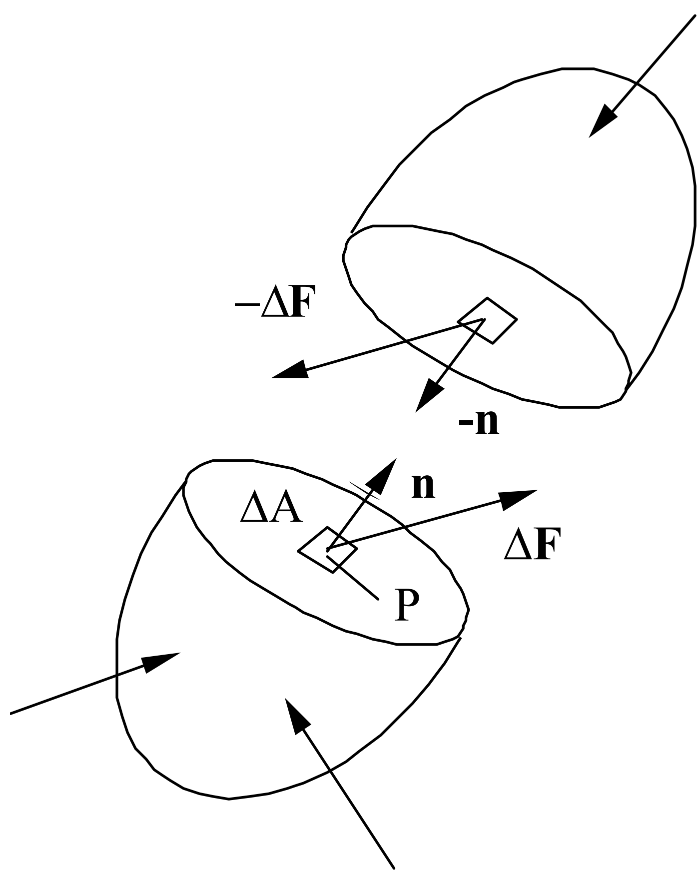
Stresses in Three Dimensions

The traction vector

The basic quantity that we use to characterize how external forces distribute themselves within a deformable body is the traction vector. This vector is defined as the force/unit area acting at a point P in the body across an imaginary cutting plane taken through the body at point P. The unit normal vector to this cutting plane is **n** (see Figure)



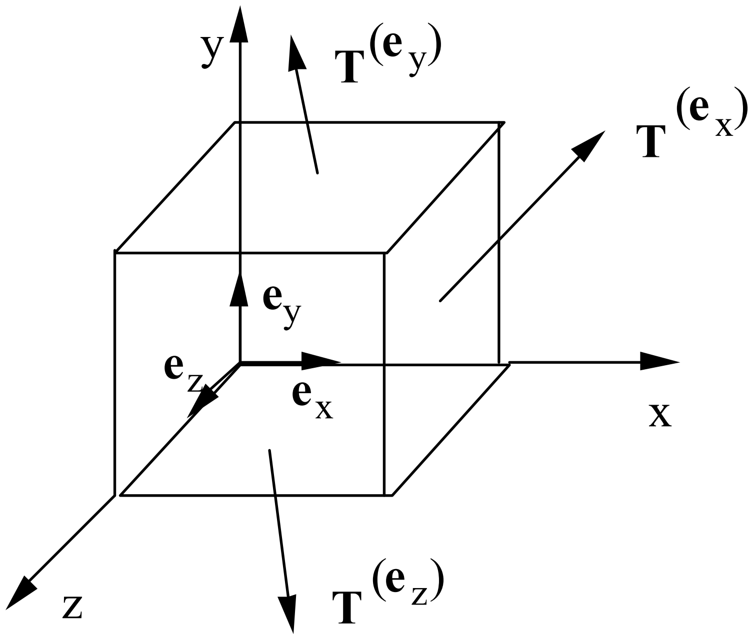
Thus, the traction vector, $\mathbf{T}^{(\mathbf{n})}$, is given by

$$\mathbf{T}^{(\mathbf{n})} = \lim_{\Delta A \to 0} \frac{\Delta \mathbf{F}}{\Delta A}$$

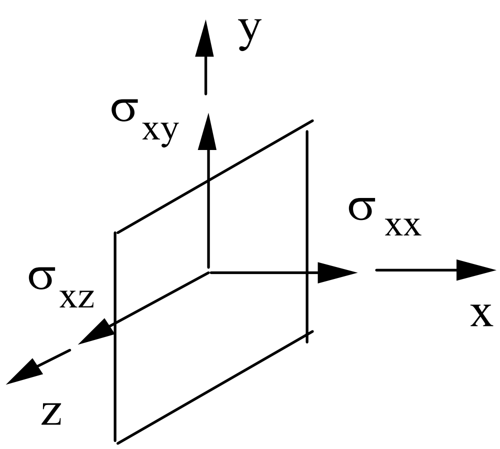
Since the internal forces always occur in equal and opposite pairs (see Figure), it follows that

$$\mathbf{T}^{(-\mathbf{n})} = -\mathbf{T}^{(\mathbf{n})}$$

If we take the cutting planes at a point P along three planes whose normals are directed along the (x, y, z) axes, then there are three traction vectors on these planes, which we call $\mathbf{T}^{(\mathbf{e}_x)}, \mathbf{T}^{(\mathbf{e}_y)}, \mathbf{T}^{(\mathbf{e}_z)}$



The components of these three traction vectors are what we call the stresses at point P with respect to the (x, y, z) axes. For example, $\sigma_{xx}, \sigma_{xy}, \sigma_{xz}$ are the components of the traction vector $\mathbf{T}^{(\mathbf{e}_x)}$



so that we have

$$\mathbf{T}^{(\mathbf{e}_x)} = \sigma_{xx}\mathbf{e}_x + \sigma_{xy}\mathbf{e}_y + \sigma_{xz}\mathbf{e}_z$$

If we similarly break down the other traction vectors acting on the y and z planes into their components, we have

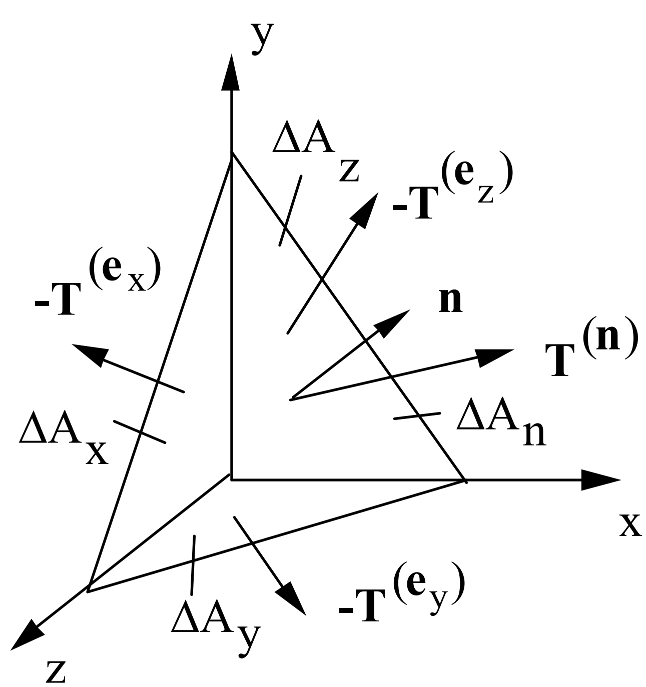
$$\mathbf{T}^{(\mathbf{e}_x)} = \sigma_{xx}\mathbf{e}_x + \sigma_{xy}\mathbf{e}_y + \sigma_{xz}\mathbf{e}_z$$
$$\mathbf{T}^{(\mathbf{e}_y)} = \sigma_{yx}\mathbf{e}_x + \sigma_{yy}\mathbf{e}_y + \sigma_{yz}\mathbf{e}_z$$
$$\mathbf{T}^{(\mathbf{e}_z)} = \sigma_{zx}\mathbf{e}_x + \sigma_{zy}\mathbf{e}_y + \sigma_{zz}\mathbf{e}_z$$

which we can write more compactly as

$$\mathbf{T}^{(\mathbf{e}_i)} = \sum_{j=1}^{3}\sigma_{ij}\mathbf{e}_j \quad (i = 1,2,3)$$

Traction vector on an arbitrary plane

The reason that these traction vectors on the x, y, z planes (and their corresponding stress components) are important is that if we know these three tractions at a point P then we can find the traction vector at point P for any other cutting plane as well. To see this consider cutting out a small tetrahedron about point P whose back faces are along the x, y, and z axes and whose front face is along an oblique plane whose unit normal is **n**:



If we require this tetrahedron to be in equilibrium then $\sum \mathbf{F} = 0$ so that we find

$$\mathbf{T}^{(\mathbf{n})}\Delta A_n - \mathbf{T}^{(\mathbf{e}_x)}\Delta A_x - \mathbf{T}^{(\mathbf{e}_y)}\Delta A_y - \mathbf{T}^{(\mathbf{e}_z)}\Delta A_z = 0$$

However, we can show that ratios of the areas of the faces of this tetrahedron are just the components of unit normal, **n**, i.e.

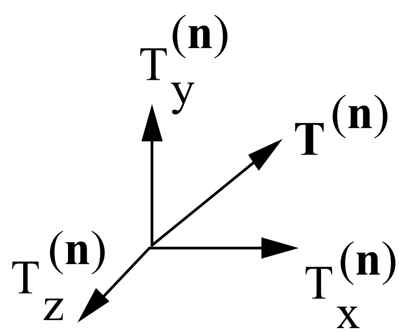
$$n_x = \frac{\Delta A_x}{\Delta A_n},\ n_y = \frac{\Delta A_y}{\Delta A_n},\ n_z = \frac{\Delta A_z}{\Delta A_n}$$

so that we find

$$\mathbf{T}^{(\mathbf{n})} = \mathbf{T}^{(\mathbf{e}_x)}n_x + \mathbf{T}^{(\mathbf{e}_y)}n_y + T^{(\mathbf{e}_z)}n_z$$

If we write the tractions on the back faces in terms of their component stresses and break up the traction vector $\mathbf{T}^{(\mathbf{n})}$ into its three components along the (x, y, z) axes, i.e.

$$\mathbf{T}^{(\mathbf{n})} = T_x^{(\mathbf{n})}\mathbf{e}_x + T_y^{(\mathbf{n})}\mathbf{e}_y + T_z^{(\mathbf{n})}\mathbf{e}_z$$



then equating components we find

$$T_x^{(\mathbf{n})} = \sigma_{xx} n_x + \sigma_{yx} n_y + \sigma_{zx} n_z$$
$$T_y^{(\mathbf{n})} = \sigma_{xy} n_x + \sigma_{yy} n_y + \sigma_{zy} n_z$$
$$T_z^{(\mathbf{n})} = \sigma_{xz} n_x + \sigma_{yz} n_y + \sigma_{zz} n_z$$

or, more compactly

$$T_i^{(\mathbf{n})} = \sum_{j=1}^{3} \sigma_{ji} n_j \quad (i = 1, 2, 3)$$

We can also write this in matrix notation if we let

$$\left\{\mathbf{T}^{(\mathbf{n})}\right\} = \left\{T_1^{(\mathbf{n})} \quad T_2^{(\mathbf{n})} \quad T_3^{(\mathbf{n})}\right\}, \quad \{\mathbf{n}\} = \{n_1 \quad n_2 \quad n_3\}, \quad [\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

so that we have

$$\left\{\mathbf{T}^{(\mathbf{n})}\right\} = \{\mathbf{n}\}[\sigma]$$

or, equivalently, since $[\sigma] = [\sigma]^T$

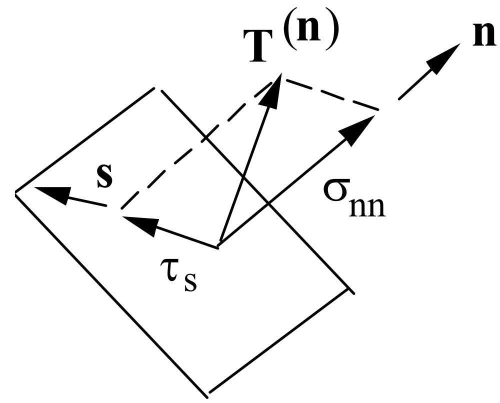
$$\left\{\mathbf{T}^{(\mathbf{n})}\right\}^T = [\sigma]\{\mathbf{n}\}^T$$

where

$$\left\{\mathbf{T}^{(\mathbf{n})}\right\}^T = \begin{Bmatrix} T_1^{(n)} \\ T_2^{(n)} \\ T_3^{(n)} \end{Bmatrix}, \quad \{\mathbf{n}\}^T = \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}$$

Normal stress and total shear stress on an oblique plane

We can always break the traction vector on an arbitrary oblique plane into two components – one along the unit normal, **n**, and the other component in the plane on which the traction acts. We will call these two components the normal stress σ_{nn} and total shear stress, τ_s, respectively and let **s** be a unit vector that defines the direction of this total shear component in the plane:



Since σ_{nn} is just the Cartesian component of $\mathbf{T}^{(\mathbf{n})}$ in the **n** direction we have

$$\begin{aligned}\sigma_{nn} &= \mathbf{T}^{(\mathbf{n})} \cdot \mathbf{n} \\ &= T_x^{(\mathbf{n})} n_x + T_y^{(\mathbf{n})} n_y + T_z^{(\mathbf{n})} n_z\end{aligned}$$

If we write the traction vector components in terms of the stresses, then we find

$$\begin{aligned}\sigma_{nn} = &\sigma_{xx} n_x^2 + \sigma_{yy} n_y^2 + \sigma_{zz} n_z^2 \\ &+2\sigma_{xy} n_x n_y + 2\sigma_{xz} n_x n_z + 2\sigma_{yz} n_y n_z\end{aligned}$$

which can also be written as

$$\sigma_{nn} = \sum_{i=1}^{3}\sum_{j=1}^{3} \sigma_{ij} n_i n_j$$

or, equivalently

$$\sigma_{nn} = \{\mathbf{n}\}[\sigma]\{\mathbf{n}\}^T$$

To obtain the total shear stress we note that its magnitude can be found since

$$\begin{aligned}|\tau_s| &= \sqrt{\left|\mathbf{T}^{(\mathbf{n})}\right|^2 - \sigma_{nn}^2} \\ &= \sqrt{\left(T_x^{(\mathbf{n})}\right)^2 + \left(T_y^{(\mathbf{n})}\right)^2 + \left(T_z^{(\mathbf{n})}\right)^2 - \sigma_{nn}^2}\end{aligned}$$

To find the direction of the total shear stress, we let

$$\mathbf{s} = s_x \mathbf{e}_x + s_y \mathbf{e}_y + s_z \mathbf{e}_z$$

and note that

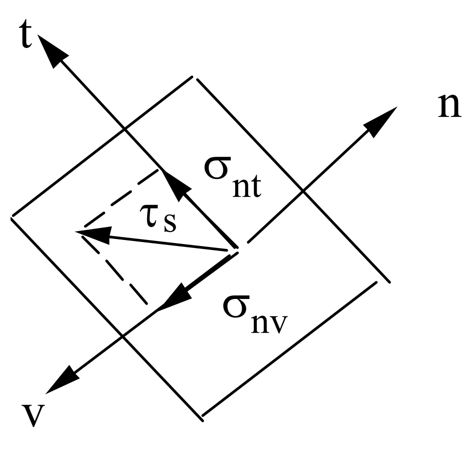
$$|\tau_s| \mathbf{s} = \mathbf{T}^{(\mathbf{n})} - \sigma_{nn} \mathbf{n}$$

which in component form leads to three equations which can be solved for s_x, s_y, s_z:

$$|\tau_s| s_x = T_x^{(\mathbf{n})} - \sigma_{nn} n_x$$
$$|\tau_s| s_y = T_y^{(\mathbf{n})} - \sigma_{nn} n_y$$
$$|\tau_s| s_z = T_z^{(\mathbf{n})} - \sigma_{nn} n_z$$

Stress components on oblique planes

The results of the last section gave us the equations we need to solve for the normal stress and total shear stress. In some cases, however, we need to know instead the individual shear stress components acting on the oblique plane along two mutually orthogonal directions that we will call t and v:



We will call these shear stress components σ_{nt}, σ_{nv} . If we let **t** and **v** be unit vectors in the t and v directions, then we can write them in terms of their components as

$$\mathbf{t} = t_x \mathbf{e}_x + t_y \mathbf{e}_y + t_z \mathbf{e}_z$$
$$\mathbf{v} = v_x \mathbf{e}_x + v_y \mathbf{e}_y + v_z \mathbf{e}_z$$

Just as we obtained the normal stress from the traction vector, we can obtain these shear stress components through

$$\sigma_{nt} = \mathbf{T}^{(\mathbf{n})} \cdot \mathbf{t}$$
$$\sigma_{nv} = \mathbf{T}^{(\mathbf{n})} \cdot \mathbf{v}$$

or, in component form

$$\sigma_{nt} = T_x^{(\mathbf{n})} t_x + T_y^{(\mathbf{n})} t_y + T_z^{(\mathbf{n})} t_z$$
$$\sigma_{nv} = T_x^{(\mathbf{n})} v_x + T_y^{(\mathbf{n})} v_y + T_z^{(\mathbf{n})} v_z$$

which become, in terms of the stresses

$$\begin{aligned}\sigma_{nt} &= \sigma_{xx} n_x t_x + \sigma_{yy} n_y t_y + \sigma_{zz} n_z t_z \\ &+\sigma_{xy}\left(n_y t_x + n_x t_y\right) \\ &+\sigma_{xz}\left(n_z t_x + n_x t_z\right) \\ &+\sigma_{yz}\left(n_z t_y + n_y t_z\right)\end{aligned}$$

and

$$\begin{aligned}\sigma_{nv} &= \sigma_{xx} n_x v_x + \sigma_{yy} n_y v_y + \sigma_{zz} n_z v_z \\ &+\sigma_{xy}\left(n_y v_x + n_x v_y\right) \\ &+\sigma_{xz}\left(n_z v_x + n_x v_z\right) \\ &+\sigma_{yz}\left(n_z v_y + n_y v_z\right)\end{aligned}$$

Both of these equations can be written more compactly in the same form as we obtained previously for σ_{nn}. Summarizing all these results we have

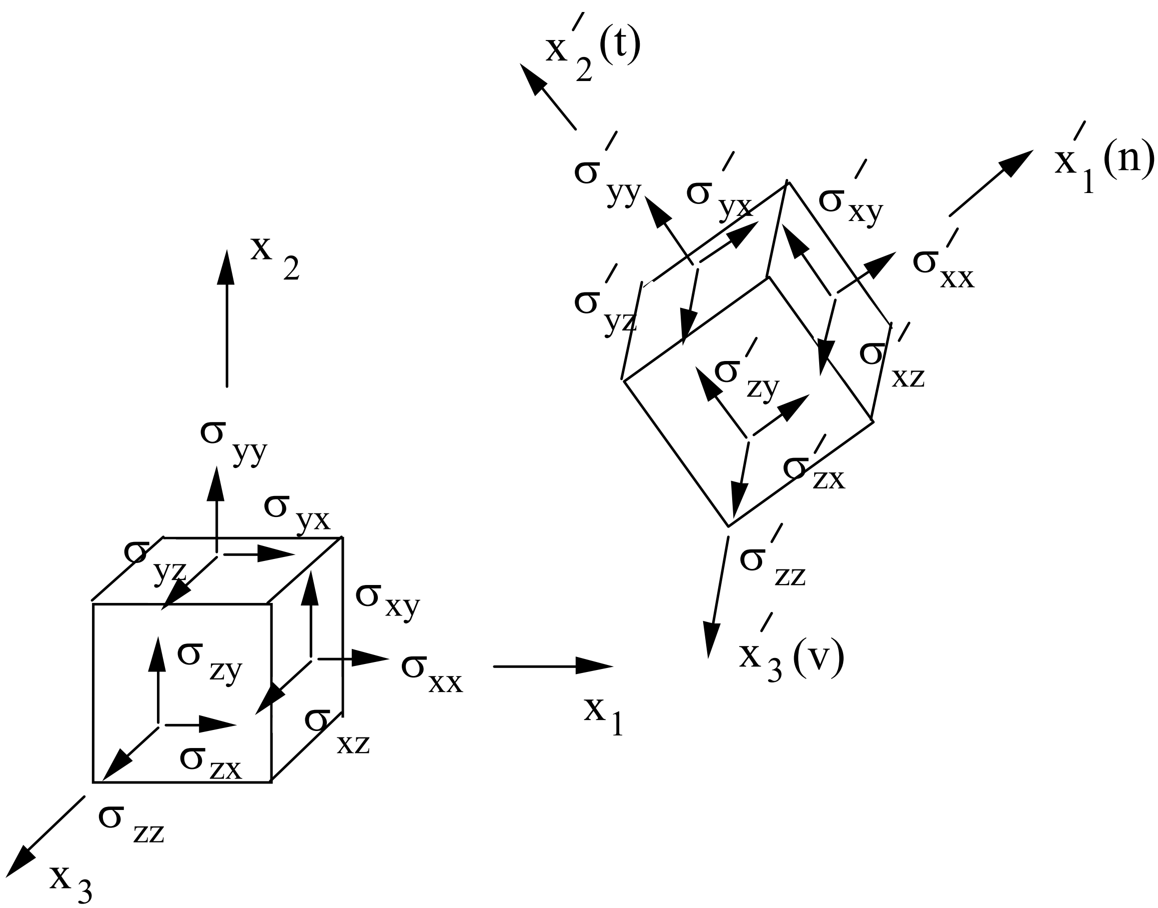
$$\sigma_{nn} = \sum_{i=1}^{3}\sum_{j=1}^{3} \sigma_{ij} n_i n_j$$
$$\sigma_{nt} = \sum_{i=1}^{3}\sum_{j=1}^{3} \sigma_{ij} n_i t_j$$
$$\sigma_{ns} = \sum_{i=1}^{3}\sum_{j=1}^{3} \sigma_{ij} n_i v_j$$

or, in matrix form

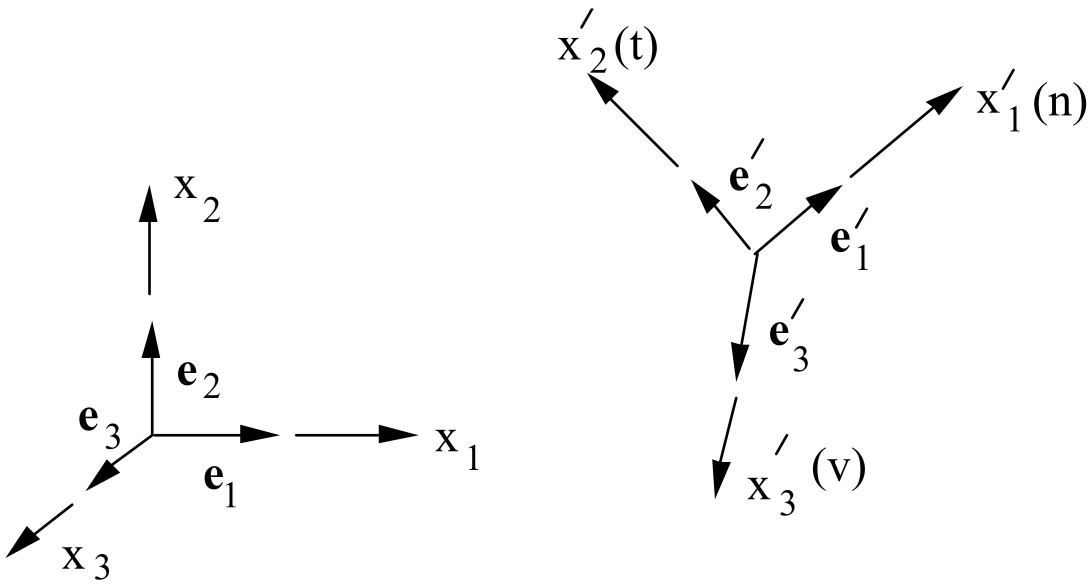
$$\sigma_{nn} = \{\mathbf{n}\}[\sigma]\{\mathbf{n}\}^T$$
$$\sigma_{nt} = \{\mathbf{n}\}[\sigma]\{\mathbf{t}\}^T$$
$$\sigma_{nv} = \{\mathbf{n}\}[\sigma]\{\mathbf{v}\}^T$$

Stress transformation equations

The results of the last section gave the results for the stress components acting on a single plane whose normal was **n**. However, since **n** was arbitrary, we can use those results, appropriately modified, to obtain the stresses acting on three mutually orthogonal oblique planes in terms of the stresses acting along the (x, y, z) planes. These relationships are called the stress transformation equations.



To obtain the stress transformation equations we note that we can identify **n**, **t**, and **v** as unit vectors along three mutually orthogonal axes (x_1', x_2', x_3') that are rotated relative to the (x, y, z) axes (see Figure). Thus, the components of **n**, **t**, and **v** relative to the (x, y, z) axes are just the direction cosines of the (x_1', x_2', x_3') axes relative to the (x, y, z) axes, i.e. if we let $(\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3')$ be unit vectors along the (x_1', x_2', x_3') axes, respectively, we have



the following:

$$\mathbf{e}_1' \equiv \mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$$
$$\mathbf{e}_2' \equiv \mathbf{t} = t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + t_3\mathbf{e}_3$$
$$\mathbf{e}_3' \equiv \mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$$

It is more convenient to write these components in a more uniform way, namely

$$n_i = l_{i1}, \quad t_i = l_{i2}, \quad v_i = l_{i3} \quad (i = 1,2,3)$$

in terms of the nine quantities $l_{ij} \quad (i = 1,2.3), \quad (j = 1,2,3)$, where l_{ij} is the direction cosine between the x_i axis and the x_j' axis. Then we find

$$\mathbf{e}_i' = \sum_{j=1}^{3} l_{ji}\mathbf{e}_j \quad (i = 1,2,3)$$

Now, consider the stress components we obtained previously $(\sigma_{nn}, \sigma_{nt}, \sigma_{nv})$. These are just the stresses in the $x_1' =$ constant plane so we have

$$\sigma'_{11} = \sigma_{nn} = \sum_{i=1}^{3}\sum_{j=1}^{3} l_{i1} l_{j1} \sigma_{ij}$$

$$\sigma'_{12} = \sigma_{nt} = \sum_{i=1}^{3}\sum_{j=1}^{3} l_{i1} l_{j2} \sigma_{ij}$$

$$\sigma'_{13} = \sigma_{nv} = \sum_{i=1}^{3}\sum_{j=1}^{3} l_{i1} l_{j3} \sigma_{ij}$$

However, if the normal to the cutting plane instead was in the x'_2 direction, we would have

$$\sigma'_{22} = \sigma_{tt} = \sum_{i=1}^{3}\sum_{j=1}^{3} l_{i2} l_{j2} \sigma_{ij}$$

$$\sigma'_{21} = \sigma_{tn} = \sum_{i=1}^{3}\sum_{j=1}^{3} l_{i2} l_{j1} \sigma_{ij}$$

$$\sigma'_{23} = \sigma_{tv} = \sum_{i=1}^{3}\sum_{j=1}^{3} l_{i2} l_{j3} \sigma_{ij}$$

and, similarly, for a cutting plane whose normal is in the x'_3 direction

$$\sigma'_{33} = \sigma_{vv} = \sum_{i=1}^{3}\sum_{j=1}^{3} l_{i3} l_{j3} \sigma_{ij}$$

$$\sigma'_{31} = \sigma_{vn} = \sum_{i=1}^{3}\sum_{j=1}^{3} l_{i3} l_{j1} \sigma_{ij}$$

$$\sigma'_{32} = \sigma_{vt} = \sum_{i=1}^{3}\sum_{j=1}^{3} l_{i3} l_{j2} \sigma_{ij}$$

Because these results are in a uniform notation, we can write them all together as

$$\sigma'_{mn} = \sum_{i=1}^{3}\sum_{j=1}^{3} l_{im} l_{jn} \sigma_{ij} \quad (m = 1,2,3), \quad (n = 1,2,3)$$

or, in matrix form

$$[\boldsymbol{\sigma}'] = [\mathbf{l}]^T [\boldsymbol{\sigma}][\mathbf{l}]$$

where

$$[\boldsymbol{\sigma}'] = \begin{bmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23} \\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33} \end{bmatrix}, \quad [\mathbf{l}] = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

Relationship between normal stress and total shear stress

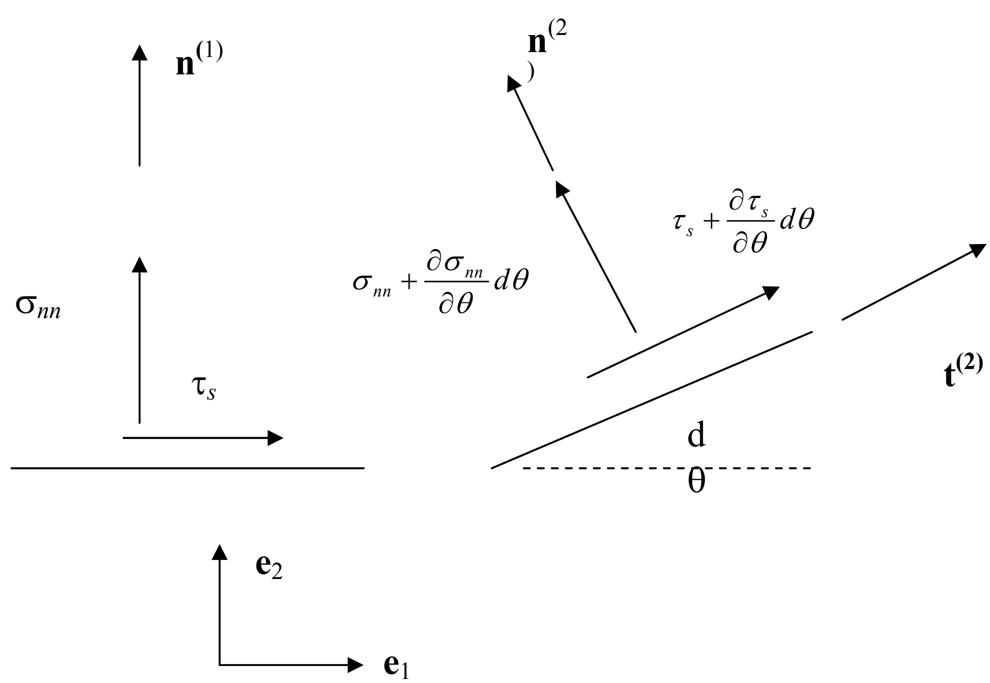
The traction vector has some properties that can tell us useful things about the relationship between normal stresses and shear stresses. For example, let $\mathbf{n}^{(1)}$ and $\mathbf{n}^{(2)}$ be the unit normals to two different planes passing through the same point. Then we have

$$\mathbf{T}^{\left(\mathbf{n}^{(1)}\right)} \cdot \mathbf{n}^{(2)} = \mathbf{T}^{\left(\mathbf{n}^{(2)}\right)} \cdot \mathbf{n}^{(1)}$$

i.e the component in the $\mathbf{n}^{(2)}$ direction of the traction vector acting on the plane whose normal is $\mathbf{n}^{(1)}$ is equal to the component in the $\mathbf{n}^{(1)}$ direction of the traction vector on the plane whose normal is $\mathbf{n}^{(2)}$. This follows since

$$\begin{aligned}\mathbf{T}^{\left(\mathbf{n}^{(1)}\right)} \cdot \mathbf{n}^{(2)} = \mathbf{T}^{\left(\mathbf{n}^{(2)}\right)} \cdot \mathbf{n}^{(1)} \\ &= \sigma_{xx} n_x^{(1)} n_x^{(2)} + \sigma_{yy} n_y^{(1)} n_y^{(2)} + \sigma_{zz} n_z^{(1)} n_z^{(2)} \\ &+ \sigma_{xy}\left(n_x^{(1)} n_y^{(2)} + n_y^{(1)} n_x^{(2)}\right) \\ &+ \sigma_{xz}\left(n_x^{(1)} n_z^{(2)} + n_z^{(1)} n_x^{(2)}\right) \\ &+ \sigma_{yz}\left(n_y^{(1)} n_z^{(2)} + n_z^{(1)} n_y^{(2)}\right)\end{aligned}$$

We can use this property as follows. Let $\mathbf{n}^{(1)}$ and $\mathbf{n}^{(2)}$ be the unit normals to two plane which are rotated slightly with respect to one another:



We have

$$\begin{aligned}\mathbf{n}^{(1)} &= \mathbf{e}_2 \\ \mathbf{n}^{(2)} &= -\sin(d\theta)\mathbf{e}_1 + \cos(d\theta)\mathbf{e}_2 \\ &\cong -d\theta\,\mathbf{e}_1 + \mathbf{e}_2 \\ \mathbf{t}^{(2)} &\cong \mathbf{e}_1 + d\theta\,\mathbf{e}_2\end{aligned}$$

so that

$$\begin{aligned}&\mathbf{T}^{\left(\mathbf{n}^{(1)}\right)} \cdot \mathbf{n}^{(2)} = \left(\tau_s \mathbf{e}_1 + \sigma_{nn}\mathbf{e}_2\right)\cdot\left(-d\theta\mathbf{e}_1 + \mathbf{e}_2\right) \\ &= -\tau_s d\theta + \sigma_{nn}\end{aligned}$$

and

$$\begin{aligned}\mathbf{T}^{\left(\mathbf{n}^{(2)}\right)} \cdot \mathbf{n}^{(1)} &= \left[\left(\sigma_{nn} + \frac{\partial \sigma_{nn}}{\partial \theta} d\theta\right)\left(-d\theta\mathbf{e}_1 + \mathbf{e}_2\right)\right]\cdot \mathbf{e}_2 \\ &+ \left[\left(\tau_s + \frac{\partial \tau_s}{\partial \theta} d\theta\right)\left(\mathbf{e}_1 + d\theta\mathbf{e}_2\right)\right]\cdot \mathbf{e}_2 \\ &= \sigma_{nn} + \frac{\partial \sigma_{nn}}{\partial \theta} d\theta + \tau_s d\theta + H.O.T.\end{aligned}$$

so that equating these two expressions we find

$$\frac{\partial \sigma_{nn}}{\partial \theta} = -2\tau_s$$

Thus, whenever the normal stress on a plane is an extremum (principal stress) so that $\frac{\partial \sigma_{nn}}{\partial \theta} = 0$ it follows that $\tau_s = 0$ on this plane also. This proves that principal planes always are free of shear stress, a result we will use in the next section to find the principal stresses and principal directions.

Principal Stresses

In the previous sections we learned how to find the tractions and stresses acting on any cutting plane. Often, however, we are interested in finding the stresses on those particular cutting planes where the normal stresses take on their extreme values. These extreme normal stresses are called principal stresses and the planes that they act on are called the principal planes. As shown in the last section, on the principal stress planes there are no shear stresses so one of the easiest ways to obtain these principal planes is to simply find the planes on which the traction vector consists of purely a normal stress, i.e.

$$\mathbf{T}^{(\mathbf{n})} = \sigma_p \mathbf{n}$$

or, equivalently, using the expression for the traction vector in terms of the stress components and the symmetry of the stresses ($\sigma_{ij} = \sigma_{ji}$)

$$T_i^{(\mathbf{n})} = \sum_{j=1}^{3} \sigma_{ij} n_j = \sigma_p n_i \quad (1 = 1,2,3)$$

where σ_p is a principal stress. Expanding these three conditions out we find a homogeneous system of equations

$$\begin{aligned}
(\sigma_{xx} - \sigma_p) n_x + \sigma_{xy} n_y + \sigma_{xz} n_z &= 0 \\
\sigma_{yx} n_x + (\sigma_{yy} - \sigma_p) n_y + \sigma_{yz} n_z &= 0 \\
\sigma_{zx} n_x + \sigma_{zy} n_y + (\sigma_{zz} - \sigma_p) n_z &= 0
\end{aligned}$$

The only way that these equations can have a nontrivial solution is for the 3x3 determinant of the coefficients to be zero, i.e.

$$\begin{vmatrix} (\sigma_{xx} - \sigma_p) & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & (\sigma_{yy} - \sigma_p) & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & (\sigma_{zz} - \sigma_p) \end{vmatrix} = 0$$

When expanded out, this condition leads to a cubic equation which has three real roots $\sigma_{p1}, \sigma_{p2}, \sigma_{p3}$ which are the principal stresses. This cubic equation can be written as

$$\sigma_p^3 - I_1 \sigma_p^2 + I_2 \sigma_p - I_3 = 0$$

where I_1, I_2, I_3 are stress invariants given by

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$
$$I_2 = \sigma_{xx}\sigma_{yy} + \sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz} - \sigma_{xy}^2 - \sigma_{xz}^2 - \sigma_{yz}^2$$
$$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{xz}\sigma_{yz} - \sigma_{xx}\sigma_{zy}^2 - \sigma_{yy}\sigma_{xz}^2 - \sigma_{zz}\sigma_{xy}^2$$

The last two invariants can also be written in more compact form using determinants as

$$I_2 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix}$$
$$I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}$$

Note that the values of these three invariants must, by definition be the same when expressed in terms of stresses as measured in <u>any</u> coordinate system. Thus, in particular, in terms of principal stresses we must have

$$I_1 = \sigma_{p1} + \sigma_{p2} + \sigma_{p3}$$
$$I_2 = \sigma_{p1}\sigma_{p2} + \sigma_{p1}\sigma_{p3} + \sigma_{p2}\sigma_{p3}$$
$$I_3 = \sigma_{p1}\sigma_{p2}\sigma_{p3}$$

After solving for the principal stresses, one check on whether our solution is error free is to compute these invariants to make sure they have the same values as originally calculated from the stresses with respect to the (x, y, z) axes.

Once the three roots of the cubic equation are found, for each of these values the original homogeneous equations can be solved for the corresponding unit normals to the principal planes given by $\left(n_x^{(1)}, n_y^{(1)}, n_z^{(1)}\right), \left(n_x^{(2)}, n_y^{(2)}, n_z^{2}\right) \left(n_x^{(3)}, n_y^{(3)}, n_z^{(3)}\right)$, where since these are unit vectors we must have

$$\left(n_x^{(1)}\right)^2 + \left(n_y^{(1)}\right)^2 + \left(n_z^{(1)}\right)^2 = 1$$
$$\left(n_x^{(2)}\right)^2 + \left(n_y^{(2)}\right)^2 + \left(n_z^{(2)}\right)^2 = 1$$
$$\left(n_x^{(3)}\right)^2 + \left(n_y^{(3)}\right)^2 + \left(n_z^{(3)}\right)^2 = 1$$

Another way to obtain the same results is to note that the solution to our original set of three homogeneous equations is really the solution to an eigenvalue problem where the principal stresses are the eigenvalues of the stress matrix

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$$

and the corresponding three unit normals are the eigenvectors of this matrix. Many commercial software packages such as MATLAB, MathCad, Mathematica, Maple, etc. have built-in functions for solving such eigenvalue problems, making the process of finding principal stresses and principal directions very easy.

We know from the previous section that the principal stresses are extreme values of the normal stress, but we did not use that fact in obtaining the principal stresses and principal planes. Here, we will show that the principal stresses and principal stress directions also can be obtained by directly requiring that the normal stress be an extremum.

Recall we found that the normal stress on any cutting plane was given by

$$\sigma_{nn} = \sum_{i=1}^{3} \sum_{j=1}^{3} n_i n_j \sigma_{ij} \qquad (1)$$

so that we see $\sigma_{nn} = \sigma_{nn}(n_1, n_2, n_3)$, i.e. the normal stress can be considered a function of the three components of the unit normal. These three components are not all independent since we must have

$$\sum_{i=1}^{3} n_i n_i = 1 \qquad (2)$$

Thus, this is an example of a constrained optimization problem where we seek the extreme values of the normal stress of Eq. (1) subject to the constraint of Eq. (2). One effective way to solve such problems is with the use of Lagrange multipliers. By this method we form up a new function, *F*, which incorporates the constraint explicitly, namely

$$F = F(n_1, n_2, n_3, \lambda) = \sum_{i=1}^{3} \sum_{j=1}^{3} n_i n_j \sigma_{ij} + \lambda \left(\sum_{i=1}^{3} n_i n_i - 1 \right)$$

where λ is a constant called a Lagrange multiplier. We then can view *F* as an unconstrained function whose extreme values are determined by setting

$$\frac{\partial F}{\partial n_1} = \frac{\partial F}{\partial n_2} = \frac{\partial F}{\partial n_3} = \frac{\partial F}{\partial \lambda} = 0$$

which gives the four conditions

$$\sum_{j=1}^{3} \sigma_{ij} n_j = \lambda n_i \quad (i = 1,2,3)$$
$$\sum_{i=1}^{3} n_i n_i = 1$$

These equations, however are identical to those we solved before if we simply identify the Lagrange multiplier with the principal stress, showing that we did indeed obtain extreme values of the normal stress.

Stresses on the octahedral plane

In addition to the principal planes and planes of extreme shear, a particular cutting plane of interest in some material failure theories is the octahedral plane, which is by definition a plane which makes equal angles with respect to the three principal directions. Thus, if we let n_j be components of the unit normal as measured with respect to the principal directions, then

$$n_1 = n_2 = n_3 = \frac{1}{\sqrt{3}}$$

Similarly, the three components of the traction vector on the octahedral plane (as measured also with respect to the principal directions) and the normal stress on the octahedral plane are given by

$$\begin{aligned} T_1 &= \sigma_{p1} n_1 \\ T_2 &= \sigma_{p2} n_2 \\ T_3 &= \sigma_{p3} n_3 \\ \sigma_{nn} &= \sigma_{p1} n_1^2 + \sigma_{p2} n_2^2 + \sigma_{p3} n_3^2 \end{aligned}$$

so that we find

$$\begin{aligned} (\sigma_{nn})_{oct} &= \frac{\sigma_{p1} + \sigma_{p2} + \sigma_{p3}}{3} = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} \\ |\tau_s|_{oct} &= \sqrt{(\sigma_{p1} n_1)^2 + (\sigma_{p2} n_2)^2 + (\sigma_{p3} n_3)^2 - (\sigma_{p1} n_1^2 + \sigma_{p2} n_2^2 + \sigma_{p3} n_3^2)^2} \\ &= \frac{1}{3}\sqrt{(\sigma_{p1} - \sigma_{p2})^2 + (\sigma_{p2} - \sigma_{p3})^2 + (\sigma_{p1} - \sigma_{p3})^2} \\ &= \frac{1}{3}\sqrt{2I_1^2 - 6I_2} \end{aligned}$$

Since the last expression for the total shear stress on the octahedral plane is in terms of the stress invariants, we can also write this total shear stress in terms of the stresses along the (x, y, z) axes. We find

$$\left|\tau_s\right|_{oct} = \frac{1}{3}\sqrt{\left(\sigma_{xx}-\sigma_{yy}\right)^2+\left(\sigma_{xx}-\sigma_{zz}\right)^2+\left(\sigma_{yy}-\sigma_{zz}\right)^2+6\left(\sigma_{xy}^2+\sigma_{yz}^2+\sigma_{xz}^2\right)}$$