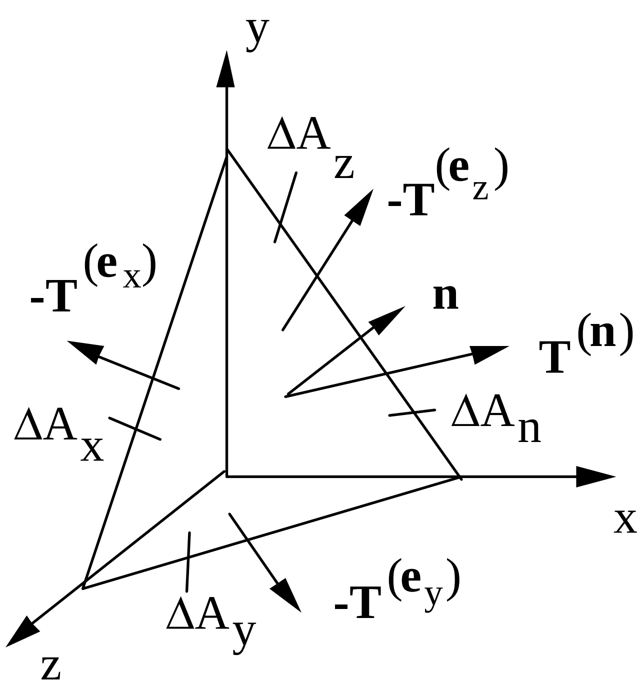


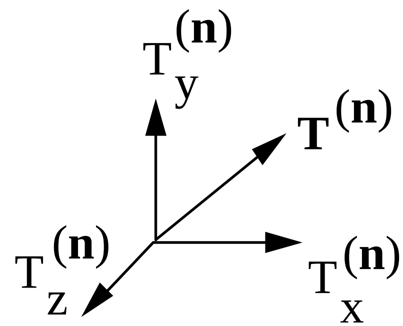
Stresses on x-y-z planes

$$\mathbf{T}^{(\mathbf{e}_x)} = \sigma_{xx}\mathbf{e}_x + \sigma_{xy}\mathbf{e}_y + \sigma_{xz}\mathbf{e}_z$$

$$\mathbf{T}^{(\mathbf{e}_y)} = \sigma_{yx}\mathbf{e}_x + \sigma_{yy}\mathbf{e}_y + \sigma_{yz}\mathbf{e}_z$$

$$\mathbf{T}^{(\mathbf{e}_z)} = \sigma_{zx}\mathbf{e}_x + \sigma_{zy}\mathbf{e}_y + \sigma_{zz}\mathbf{e}_z$$

x-y-z traction vector components



$$T_x^{(\mathbf{n})} = \sigma_{xx}n_x + \sigma_{yx}n_y + \sigma_{zx}n_z$$

$$T_y^{(\mathbf{n})} = \sigma_{xy}n_x + \sigma_{yy}n_y + \sigma_{zy}n_z$$

$$T_z^{(\mathbf{n})} = \sigma_{xz}n_x + \sigma_{yz}n_y + \sigma_{zz}n_z$$

traction vector components

$$\left\{\mathbf{T}^{(\mathbf{n})}\right\} = \left\{T_1^{(\mathbf{n})} \quad T_2^{(\mathbf{n})} \quad T_3^{(\mathbf{n})}\right\}, \quad \{\mathbf{n}\} = \{n_1 \quad n_2 \quad n_3\}, \quad [\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

$$\left\{\mathbf{T}^{(\mathbf{n})}\right\} = \{\mathbf{n}\}[\sigma]$$

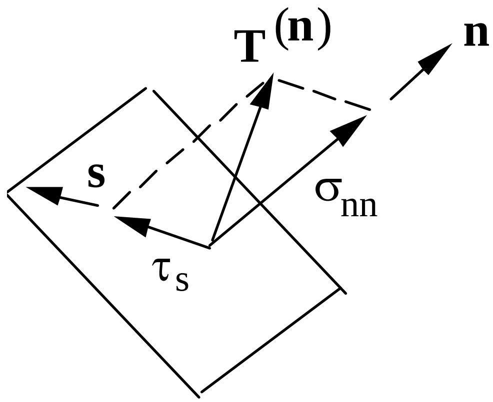
$$\left\{T_x^{(n)} \quad T_y^{(n)} \quad T_z^{(n)}\right\} = \{n_x \quad n_y \quad n_z\} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

because the stress matrix is symmetric we can also write

$$\left\{\mathbf{T}^{(\mathbf{n})}\right\}^T = [\sigma]\{\mathbf{n}\}^T \qquad \left\{\mathbf{T}^{(\mathbf{n})}\right\}^T = \begin{Bmatrix} T_1^{(n)} \\ T_2^{(n)} \\ T_3^{(n)} \end{Bmatrix}, \quad \{\mathbf{n}\}^T = \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}$$

$$\begin{Bmatrix} T_x^{(n)} \\ T_y^{(n)} \\ T_z^{(n)} \end{Bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$$

stresses on an arbitrary cutting plane



normal stress

$$\sigma_{nn} = \mathbf{T}^{(\mathbf{n})} \cdot \mathbf{n}$$
$$= T_x^{(\mathbf{n})} n_x + T_y^{(\mathbf{n})} n_y + T_z^{(\mathbf{n})} n_z$$

$$\sigma_{nn} = \sigma_{xx} n_x^2 + \sigma_{yy} n_y^2 + \sigma_{zz} n_z^2 + 2\sigma_{xy} n_x n_y + 2\sigma_{xz} n_x n_z + 2\sigma_{yz} n_y n_z$$

$$\sigma_{nn} = \{\mathbf{n}\}[\sigma]\{\mathbf{n}\}^T$$

total shear stress

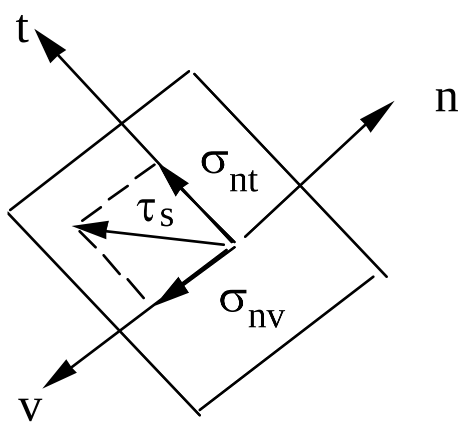
$$|\tau_s| = \sqrt{\left|\mathbf{T}^{(\mathbf{n})}\right|^2 - \sigma_{nn}^2}$$
$$= \sqrt{\left(T_x^{(\mathbf{n})}\right)^2 + \left(T_y^{(\mathbf{n})}\right)^2 + \left(T_z^{(\mathbf{n})}\right)^2 - \sigma_{nn}^2}$$

$$\mathbf{s} = s_x \mathbf{e}_x + s_y \mathbf{e}_y + s_z \mathbf{e}_z$$

$$|\tau_s| \mathbf{s} = \mathbf{T}^{(\mathbf{n})} - \sigma_{nn} \mathbf{n}$$

$$|\tau_s| s_x = T_x^{(\mathbf{n})} - \sigma_{nn} n_x$$
$$|\tau_s| s_y = T_y^{(\mathbf{n})} - \sigma_{nn} n_y$$
$$|\tau_s| s_z = T_z^{(\mathbf{n})} - \sigma_{nn} n_z$$

stresses on an arbitrary cutting plane



$$\mathbf{t} = t_x \mathbf{e}_x + t_y \mathbf{e}_y + t_z \mathbf{e}_z$$

$$\mathbf{v} = v_x \mathbf{e}_x + v_y \mathbf{e}_y + v_z \mathbf{e}_z$$

$$\sigma_{nt} = \mathbf{T}^{(\mathbf{n})} \cdot \mathbf{t} \qquad \sigma_{nt} = T_x^{(\mathbf{n})} t_x + T_y^{(\mathbf{n})} t_y + T_z^{(\mathbf{n})} t_z$$

$$\sigma_{nv} = \mathbf{T}^{(\mathbf{n})} \cdot \mathbf{v} \qquad \sigma_{nv} = T_x^{(\mathbf{n})} v_x + T_y^{(\mathbf{n})} v_y + T_z^{(\mathbf{n})} v_z$$

$$\begin{aligned}\sigma_{nt} = {} & \sigma_{xx} n_x t_x + \sigma_{yy} n_y t_y + \sigma_{zz} n_z t_z \\ & + \sigma_{xy}\left(n_y t_x + n_x t_y\right) \\ & + \sigma_{xz}\left(n_z t_x + n_x t_z\right) \\ & + \sigma_{yz}\left(n_z t_y + n_y t_z\right)\end{aligned}$$

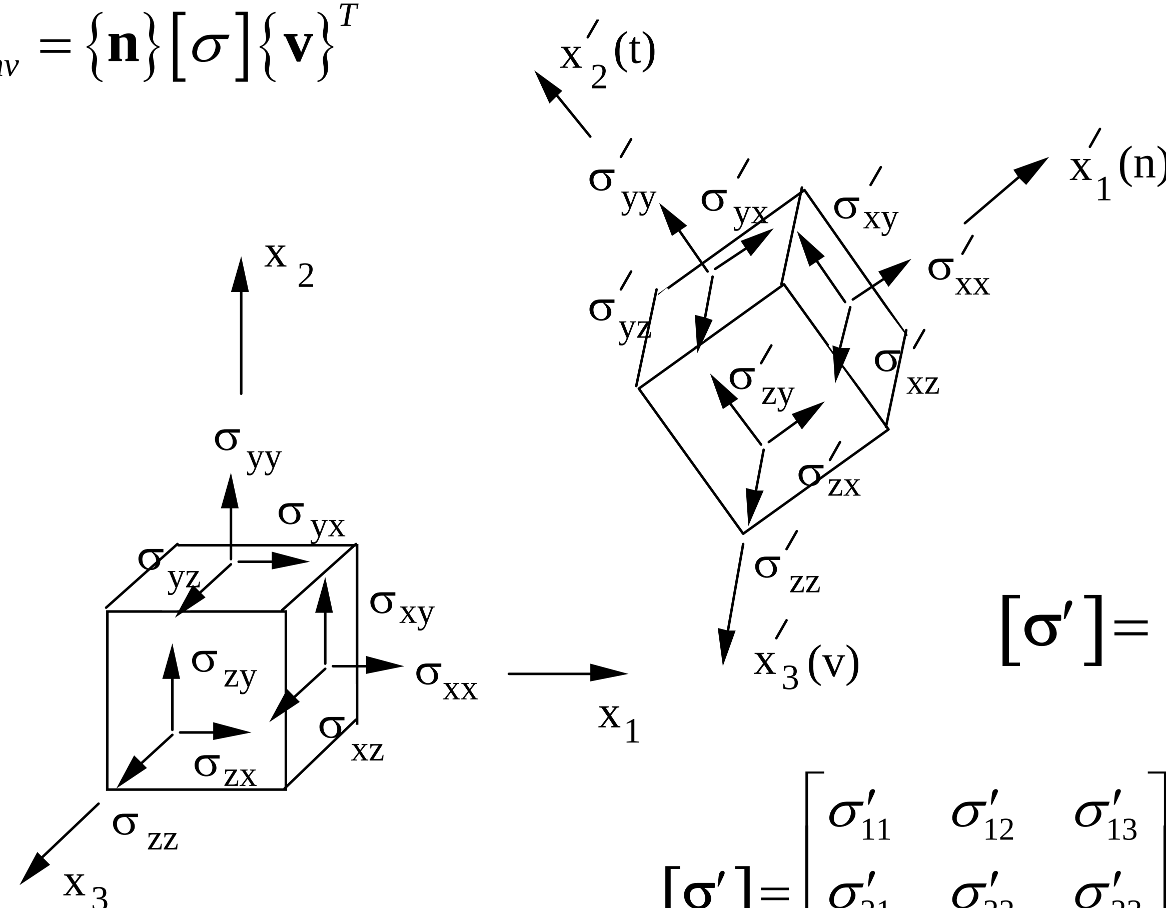
$$\begin{aligned}\sigma_{nv} = {} & \sigma_{xx} n_x v_x + \sigma_{yy} n_y v_y + \sigma_{zz} n_z v_z \\ & + \sigma_{xy}\left(n_y v_x + n_x v_y\right) \\ & + \sigma_{xz}\left(n_z v_x + n_x v_z\right) \\ & + \sigma_{yz}\left(n_z v_y + n_y v_z\right)\end{aligned}$$

$$\sigma_{nn} = \{\mathbf{n}\}[\sigma]\{\mathbf{n}\}^T$$

$$\sigma_{nt} = \{\mathbf{n}\}[\sigma]\{\mathbf{t}\}^T$$

$$\sigma_{nv} = \{\mathbf{n}\}[\sigma]\{\mathbf{v}\}^T$$

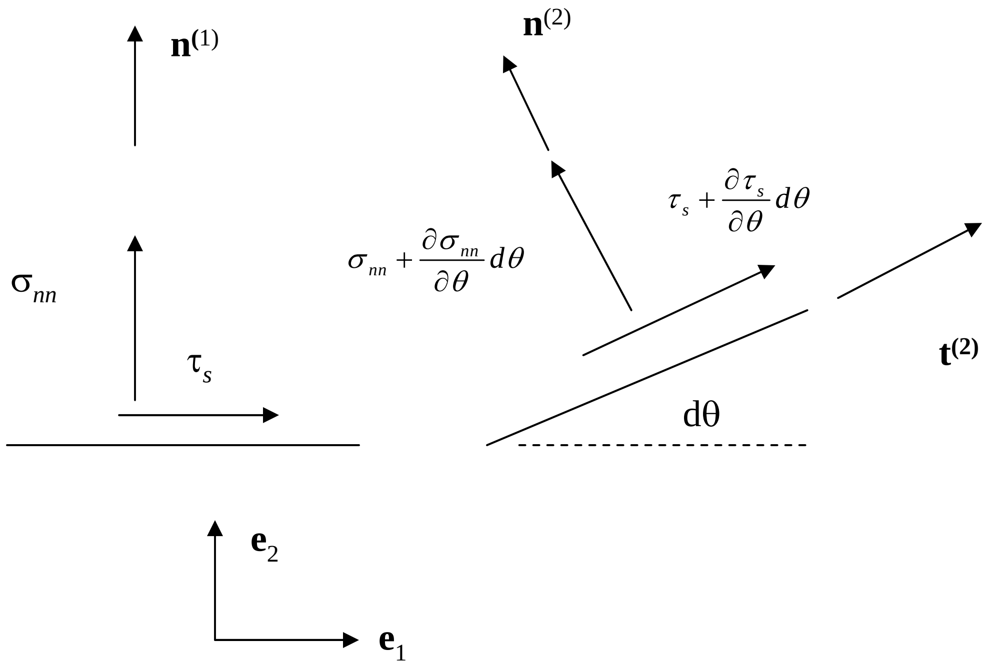
stress transformation relations



$$[\boldsymbol{\sigma}'] = [\mathbf{l}]^T[\boldsymbol{\sigma}][\mathbf{l}]$$

$$[\boldsymbol{\sigma}'] = \begin{bmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23} \\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33} \end{bmatrix}, \quad [\mathbf{l}] = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

relationship between changes of normal stress and the total shear stress



$$\frac{\partial \sigma_{nn}}{\partial \theta} = -2\tau_s$$

Principal stresses

no shear stresses on planes of extreme normal stress

$$\mathbf{T}^{(\mathbf{n})} = \sigma_p \mathbf{n}$$

homogeneous system

$$\left(\sigma_{xx} - \sigma_p\right) n_x + \sigma_{xy} n_y + \sigma_{xz} n_z = 0$$

$$\sigma_{yx} n_x + \left(\sigma_{yy} - \sigma_p\right) n_y + \sigma_{yz} n_z = 0$$

$$\sigma_{zx} n_x + \sigma_{zy} n_y + \left(\sigma_{zz} - \sigma_p\right) n_z = 0$$

for non-trivial solution

$$\begin{vmatrix} \left(\sigma_{xx} - \sigma_p\right) & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \left(\sigma_{yy} - \sigma_p\right) & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \left(\sigma_{zz} - \sigma_p\right) \end{vmatrix} = 0$$

cubic

$$\sigma_p^3 - I_1\sigma_p^2 + I_2\sigma_p - I_3 = 0$$

stress invariants

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$I_2 = \sigma_{xx}\sigma_{yy} + \sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz} - \sigma_{xy}^2 - \sigma_{xz}^2 - \sigma_{yz}^2$$

$$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{xz}\sigma_{yz} - \sigma_{xx}\sigma_{zy}^2 - \sigma_{yy}\sigma_{xz}^2 - \sigma_{zz}\sigma_{xy}^2$$

$$I_2 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix}$$

$$I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}$$

$$I_1 = \sigma_{p1} + \sigma_{p2} + \sigma_{p3}$$

$$I_2 = \sigma_{p1}\sigma_{p2} + \sigma_{p1}\sigma_{p3} + \sigma_{p2}\sigma_{p3}$$

$$I_3 = \sigma_{p1}\sigma_{p2}\sigma_{p3}$$

To solve for the principal directions

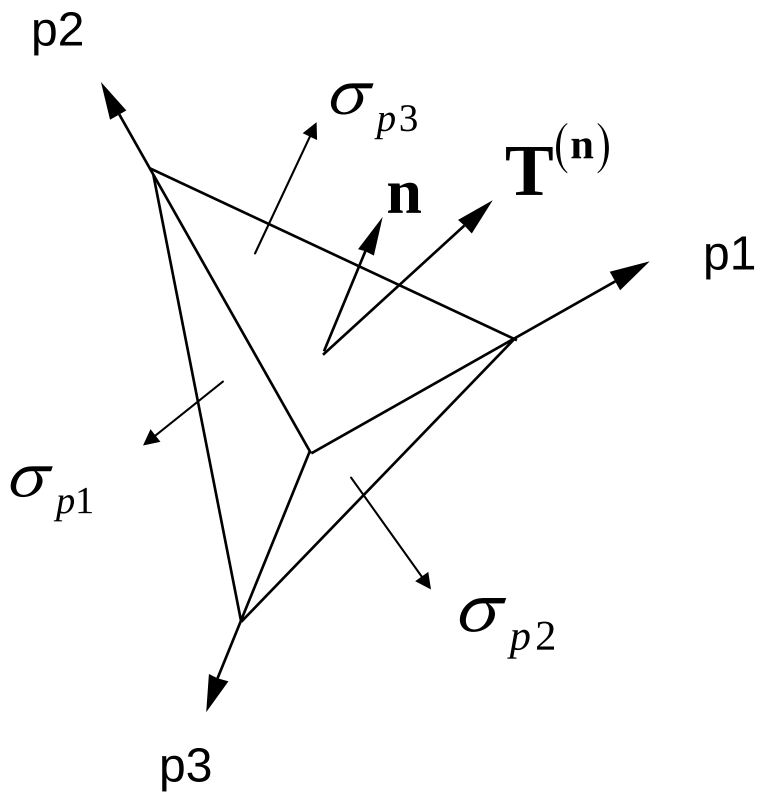
$$\left(\sigma_{xx} - \sigma_p\right) n_x + \sigma_{xy} n_y + \sigma_{xz} n_z = 0$$

$$\sigma_{yx} n_x + \left(\sigma_{yy} - \sigma_p\right) n_y + \sigma_{yz} n_z = 0$$

$$\sigma_{zx} n_x + \sigma_{zy} n_y + \left(\sigma_{zz} - \sigma_p\right) n_z = 0$$

$$\left(n_x\right)^2 + \left(n_y\right)^2 + \left(n_z\right)^2 = 1$$

stresses on the octahedral plane(s)



$$n_1 = n_2 = n_3 = \frac{1}{\sqrt{3}}$$

$$T_1 = \sigma_{p1} n_1$$
$$T_2 = \sigma_{p2} n_2$$
$$T_3 = \sigma_{p3} n_3$$
$$\sigma_{nn} = \sigma_{p1} n_1^2 + \sigma_{p2} n_2^2 + \sigma_{p3} n_3^2$$

$$(\sigma_{nn})_{oct} = \frac{\sigma_{p1} + \sigma_{p2} + \sigma_{p3}}{3} = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3}$$

$$|\tau_s|_{oct} = \sqrt{(\sigma_{p1} n_1)^2 + (\sigma_{p2} n_2)^2 + (\sigma_{p3} n_3)^2 - (\sigma_{p1} n_1^2 + \sigma_{p2} n_2^2 + \sigma_{p3} n_3^2)^2}$$
$$= \frac{1}{3}\sqrt{(\sigma_{p1} - \sigma_{p2})^2 + (\sigma_{p2} - \sigma_{p3})^2 + (\sigma_{p1} - \sigma_{p3})^2}$$
$$= \frac{1}{3}\sqrt{2I_1^2 - 6I_2}$$