MOHR'S CIRCLE FOR 3-D STRESSES

If we write the stresses on an arbitrary cutting plane whose unit normal is **n** in term of the principal stresses and their coordinate directions (see Fig. 1) then we have:

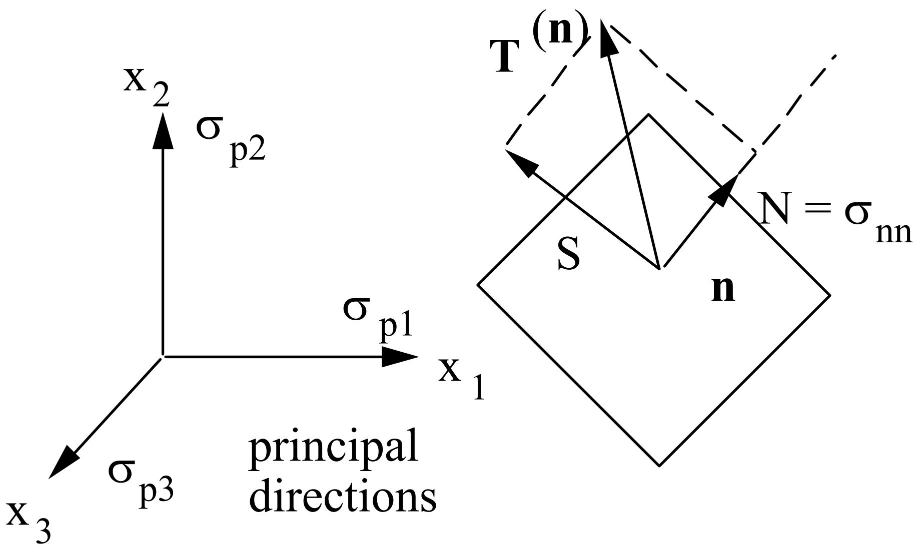

Fig. 1

the traction vector components along the principal directions as

$$T_1^{(\mathbf{n})} = n_1\sigma_{p1}$$
$$T_2^{(\mathbf{n})} = n_2\sigma_{p2}$$
$$T_3^{(\mathbf{n})} = n_3\sigma_{p3}$$

and the normal and total shear stresses given by

$$N \equiv \sigma_{nn} = \sigma_{p1}n_1^2 + \sigma_{p2}n_2^2 + \sigma_{p3}n_3^2$$
$$S^2 = \left(T_1^{(\mathbf{n})}\right)^2 + \left(T_2^{(\mathbf{n})}\right)^2 + \left(T_3^{(\mathbf{n})}\right)^2 - N^2$$
$$n_1^2 + n_2^2 + n_3^2 = 1$$

which can be considered to be three equations for the unit normal components (squared) in terms of N, S, and the principal stresses, whose solution is

$$n_1^2 = \frac{S^2 + (N-\sigma_{p2})(N-\sigma_{p3})}{(\sigma_{p1}-\sigma_{p2})(\sigma_{p1}-\sigma_{p3})} \geq 0$$
$$n_2^2 = \frac{S^2 + (N-\sigma_{p1})(N-\sigma_{p3})}{(\sigma_{p2}-\sigma_{p3})(\sigma_{p2}-\sigma_{p1})} \geq 0 \qquad (1)$$
$$n_3^2 = \frac{S^2 + (N-\sigma_{p1})(N-\sigma_{p2})}{(\sigma_{p3}-\sigma_{p1})(\sigma_{p3}-\sigma_{p2})} \geq 0$$

If we order the three principal stresses such that

$$\sigma_{p1} > \sigma_{p2} > \sigma_{p3}$$

then the inequalities of Eq. (1) imply that

$$S^2 + (N - \sigma_{p2})(N - \sigma_{p3}) \geq 0$$
$$S^2 + (N - \sigma_{p3})(N - \sigma_{p1}) \leq 0$$
$$S^2 + (N - \sigma_{p1})(N - \sigma_{p2}) \geq 0$$

which can be also rewritten equivalently as

$$S^2 + \left(N - \frac{\sigma_{p2} + \sigma_{p3}}{2} \right)^2 \geq \left(\frac{\sigma_{p2} - \sigma_{p3}}{2} \right)^2$$
$$S^2 + \left(N - \frac{\sigma_{p1} + \sigma_{p3}}{2} \right)^2 \leq \left(\frac{\sigma_{p3} - \sigma_{p1}}{2} \right)^2 \qquad (2)$$
$$S^2 + \left(N - \frac{\sigma_{p1} + \sigma_{p2}}{2} \right)^2 \geq \left(\frac{\sigma_{p1} - \sigma_{p2}}{2} \right)^2$$

If we plot the two quantities S and N, the three inequalities in Eq. (2) can be interpreted geometrically as the regions exterior or interior to three circles (see shaded region of Fig. 2)

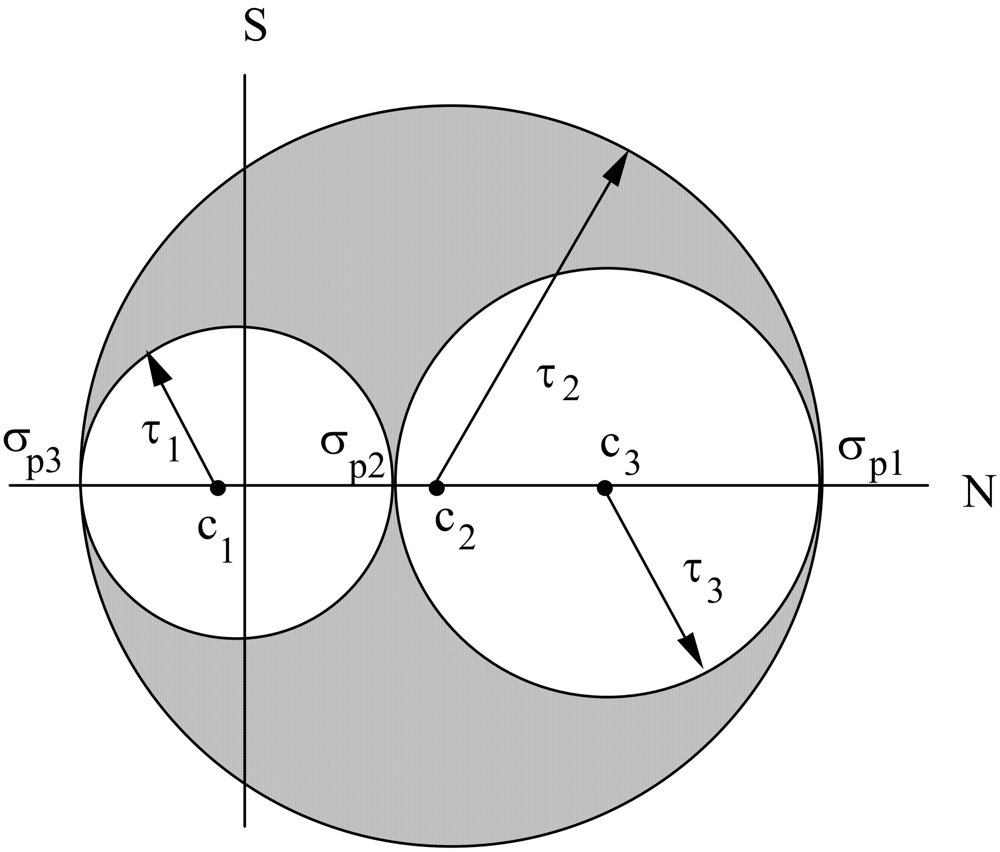

Fig.2

whose centers are at

$$c_1 = \frac{\sigma_{p2} + \sigma_{p3}}{2}$$

$$c_2 = \frac{\sigma_{p1} + \sigma_{p3}}{2}$$

$$c_3 = \frac{\sigma_{p1} + \sigma_{p2}}{2}$$

and whose radii are

$$\tau_1 = \frac{|\sigma_{p2} - \sigma_{p3}|}{2}$$
$$\tau_2 = \frac{|\sigma_{p3} - \sigma_{p1}|}{2}$$
$$\tau_3 = \frac{|\sigma_{p1} - \sigma_{p2}|}{2}$$

which are also the three extreme values of the total shear stress.

Since all the possible stresses on any cutting plane lie within the shaded region of Fig. 2 and not just on the three Mohr's circles, this 3-D figure is not too convenient to use to find stresses in general (it's better to find them directly from the stress transformation equations). However, the 3-D Mohr's circle construction is useful to locate the planes of extreme shear with respect to the principal directions. For example, we see that there are planes of extreme shear when

$$N = \frac{\sigma_{p1} + \sigma_{p2}}{2}$$
$$S^2 = \left(\frac{\sigma_{p1} - \sigma_{p2}}{2}\right)^2$$

so that from Eq. (1) we can solve for the squares of the components of the unit normal. we find:

$$n_1^2 = \frac{1}{2}, \quad n_2^2 = \frac{1}{2}, \quad n_3^2 = 0$$

Similarly, when

$$N = \frac{\sigma_{p1} + \sigma_{p3}}{2}$$
$$S^2 = \left(\frac{\sigma_{p1} - \sigma_{p3}}{2}\right)^2$$

we find

$$n_1^2 = \frac{1}{2}, \quad n_3^2 = \frac{1}{2}, \quad n_2^2 = 0$$

and finally, for

$$N = \frac{\sigma_{p2} + \sigma_{p3}}{2}$$

$$S^2 = \left(\frac{\sigma_{p2} - \sigma_{p3}}{2}\right)^2$$

we have

$$n_2^2 = \frac{1}{2}, \quad n_3^2 = \frac{1}{2}, \quad n_1^2 = 0$$

Summarizing all these results

n_1	n_2	n_3	S^2	N
0	$\pm 1/\sqrt{2}$	$\pm 1/\sqrt{2}$	$(\sigma_{p2}-\sigma_{p3})^2/4$	$(\sigma_{p2}+\sigma_{p3})/2$
$\pm 1/\sqrt{2}$	0	$\pm 1/\sqrt{2}$	$(\sigma_{p1}-\sigma_{p3})^2/4$	$(\sigma_{p1}+\sigma_{p3})/2$
$\pm 1/\sqrt{2}$	$\pm 1/\sqrt{2}$	0	$(\sigma_{p1}-\sigma_{p2})^2/4$	$(\sigma_{p1}+\sigma_{p2})/2$

which indicates that that the planes of extreme shear all lie at ±45° from the principal directions. Figure 3 shows one of these cases explicitly:

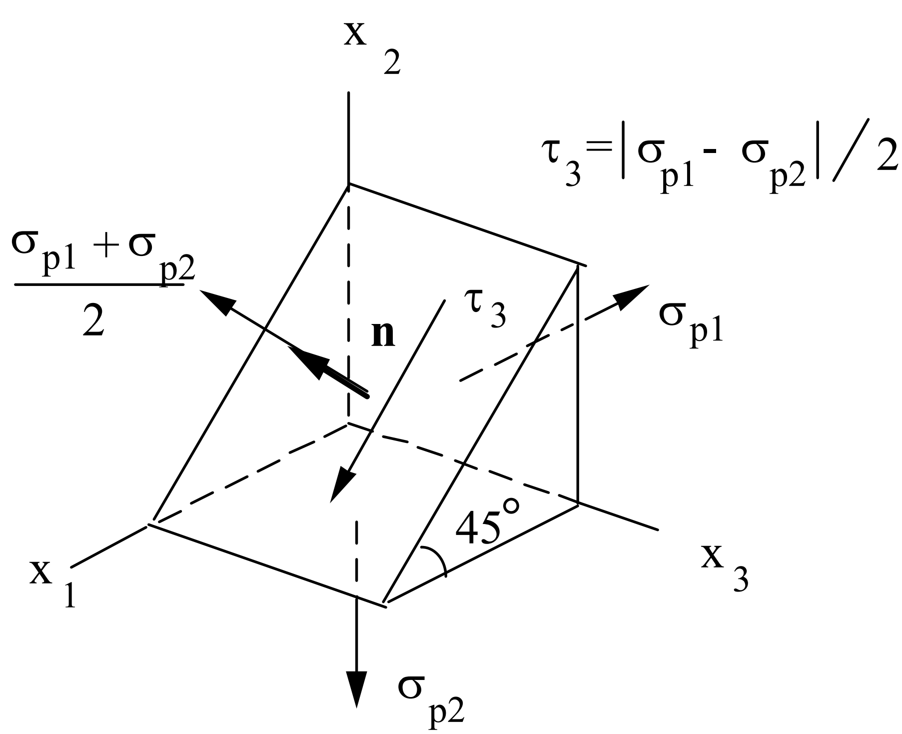

Fig. 3