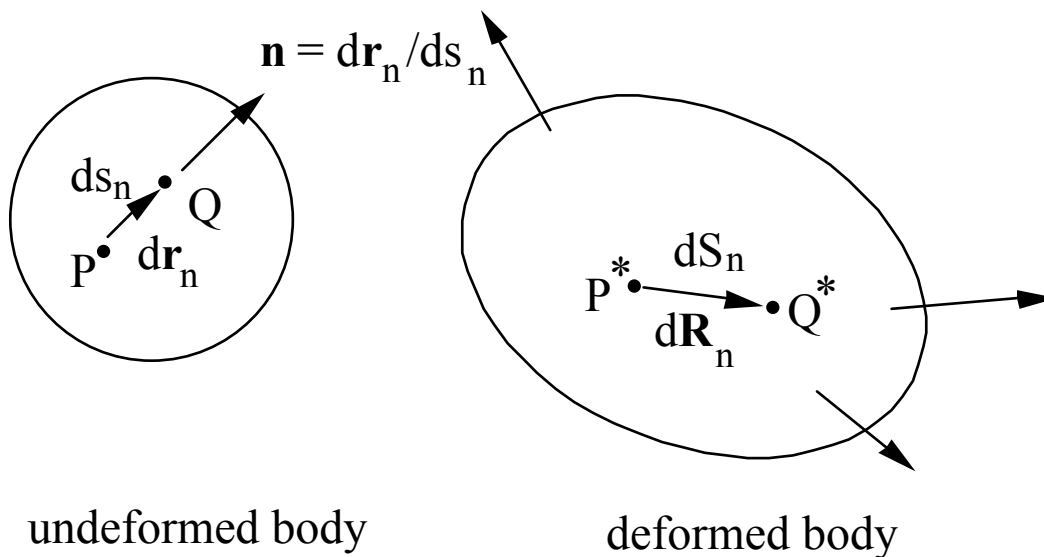


## Strains in Three Dimensions

### Definitions of normal and shear strains

When a body is loaded and deforms, elements of that body elongate or contract and are distorted. The quantity that we use to measure the changes of length occurring in the body is called the normal strain. Consider for example, a small directed line segment  $d\mathbf{r}_n$  between two very close points  $P$  and  $Q$  in the undeformed body. We will define the length of this small segment as  $ds_n$  and its direction by the unit vector  $\mathbf{n}$ . When this body is loaded and deforms, point  $P$  will move to a new point  $P^*$  and  $Q$  to a new point  $Q^*$  and



the original line segment will change to a new segment  $d\mathbf{R}_n$  of length  $dS_n$  (see Figure)

### Normal Strain

We will measure the change of length occurring for this small line segment in terms of the normal strain,  $E_{nn}$ , at point  $P$  which is defined as

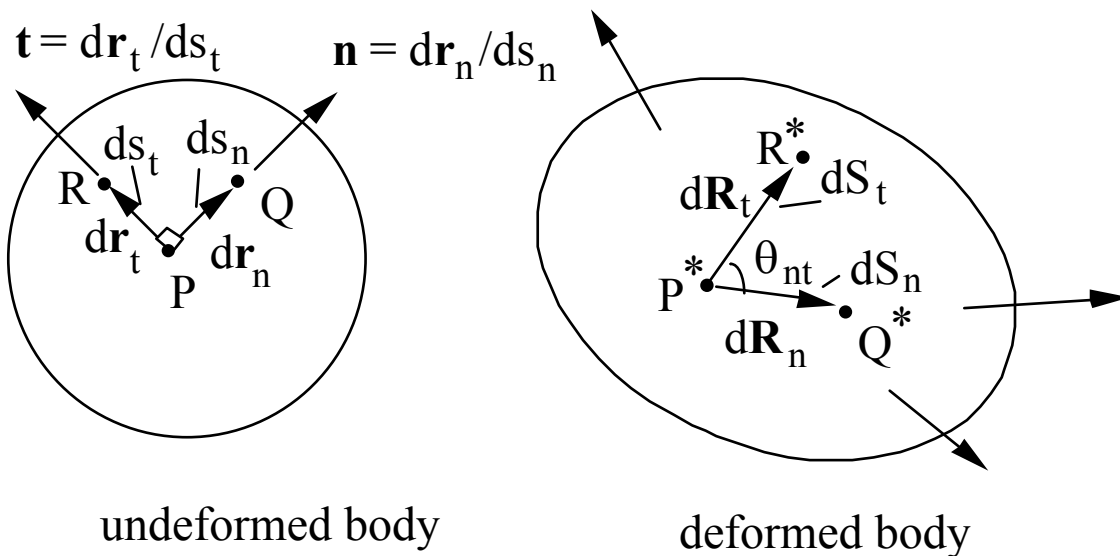
$$\begin{aligned}
 E_m &= \frac{dS_n^2 - ds_n^2}{2ds_n^2} \\
 &= \frac{d\mathbf{R}_n \cdot d\mathbf{R}_n - d\mathbf{r}_n \cdot d\mathbf{r}_n}{2ds_n^2} \\
 &= \frac{1}{2} \left( \left| \frac{d\mathbf{R}_n}{ds_n} \right|^2 - 1 \right)
 \end{aligned}$$

This definition can be used regardless of the size of the strain, so that it is a measure of the deformation that is useful in general. However, if the changes of lengths occurring in the body are very small, as often occurs in practice, this normal strain reduces to the normal small strain,  $e_m$ , defined as the relative elongation of the line segment, i.e.

$$\begin{aligned}
 E_m &= \frac{(dS_n - ds_n)(dS_n + ds_n)}{2ds_n^2} \\
 &\cong e_m = \frac{dS_n - ds_n}{ds_n}
 \end{aligned}$$

### Shear Strain

In addition to changes of length occurring in the body, there are distortions (changes of angle) which also must be characterized. The shear strain will be the quantity we use to define these changes. To obtain the shear strain we consider two line segments in the undeformed body along the  $\mathbf{n}$  and  $\mathbf{t}$  directions which initially are at right angles to each other (see Figure). In the deformed body, these line segments will no longer be orthogonal. We will let  $\theta_{nt}$  be the new angle between the two distorted line segments.



The shear strain,  $E_{nt}$ , will then be defined as

$$E_{nt} = \frac{1}{2} \frac{d\mathbf{R}_n}{ds_n} \cdot \frac{d\mathbf{R}_t}{ds_t}$$

This shear strain is related to the angular distortions occurring in the body since the cosine of the angle between two unit vectors is just the dot product of those two vectors. Thus

$$\cos \theta_{nt} = \sin\left(\frac{\pi}{2} - \theta_{nt}\right) = \frac{\frac{d\mathbf{R}_n}{ds_n} \cdot \frac{d\mathbf{R}_t}{ds_t}}{\left|\frac{d\mathbf{R}_n}{ds_n}\right| \left|\frac{d\mathbf{R}_t}{ds_t}\right|}$$

Using the definitions of the normal and shear strains for the  $\mathbf{n}$  and  $\mathbf{t}$  directions, then

$$\sin\left(\frac{\pi}{2} - \theta_{nt}\right) = \frac{2E_{nt}}{\sqrt{1+2E_{nn}}\sqrt{1+2E_{tt}}}$$

which shows that if  $\theta_{nt} = \frac{\pi}{2}$  so that there is no angular distortion of these two lines, the shear stress  $E_{nt}$  indeed is zero. When the strains are small we have

$$\begin{aligned} \sin\left(\frac{\pi}{2} - \theta_{nt}\right) &\cong \left(\frac{\pi}{2} - \theta_{nt}\right) \\ E_{nn} &\ll 1, \quad E_{tt} \ll 1 \end{aligned}$$

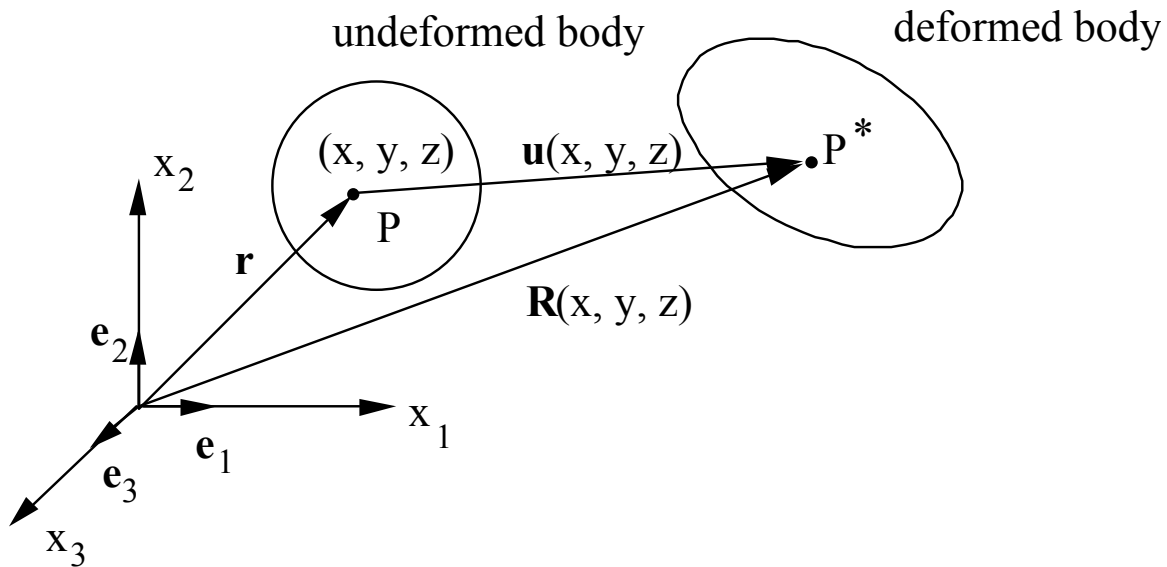
so that

$$E_{nt} \cong e_{nt} = \frac{\gamma_{nt}}{2} = \frac{1}{2} \left(\frac{\pi}{2} - \theta_{nt}\right)$$

i.e. the small strain approximation for  $E_{nt}$  is  $e_{nt}$  which is just one half the ordinary engineering shear strain,  $\gamma_{nt}$ , defined as the change of the angle (from ninety degrees) of two originally orthogonal line segments. As we will see when we talk about strain transformations, it is more convenient to use the shear strain  $e_{nt}$  rather than the engineering shear strain  $\gamma_{nt}$ .

### Strain-Displacement Relations

Although the normal strains and shear strains completely define the deformation locally everywhere in the body, knowing these strains it is not easy to directly determine what the shape of the deformed body actually is. A more convenient way to define the deformation is to describe the displacement vector,  $\mathbf{u}(x, y, z)$  for every point  $P = (x, y, z)$  in the undeformed body



Obviously the strains are related to the displacements, so we need to determine these strain-displacement relations. From the geometry we see that

$$\begin{aligned}\mathbf{R}(x_1, x_2, x_3) &= \mathbf{r} + \mathbf{u}(x_1, x_2, x_3) \\ &= \sum_{i=1}^3 x_i \mathbf{e}_i + \mathbf{u}(x_1, x_2, x_3)\end{aligned}$$

so that if  $\mathbf{r}$  changes by a small amount  $d\mathbf{r}_n$  in the  $\mathbf{n}$  direction, we have

$$d\mathbf{R}_n = d\mathbf{r}_n + d\mathbf{u}$$

or, dividing by the length of  $d\mathbf{r}_n$ ,  $ds_n$

$$\begin{aligned}\frac{d\mathbf{R}_n}{ds_n} &= \frac{d\mathbf{r}_n}{ds_n} + \frac{d\mathbf{u}}{ds_n} \\ &= \mathbf{n} + \frac{d\mathbf{u}}{ds_n}\end{aligned}$$

Thus, using this result in the definition of the normal strain we have

$$E_{nn} = \mathbf{n} \cdot \frac{d\mathbf{u}}{ds_n} + \frac{1}{2} \frac{d\mathbf{u}}{ds_n} \cdot \frac{d\mathbf{u}}{ds_n}$$

Similarly, if we let  $\mathbf{r}$  change by a small amount  $d\mathbf{r}_t$  in the  $\mathbf{t}$  direction we have

$$\begin{aligned}\frac{d\mathbf{R}_t}{ds_t} &= \frac{d\mathbf{r}_t}{ds_t} + \frac{d\mathbf{u}}{ds_t} \\ &= \mathbf{t} + \frac{d\mathbf{u}}{ds_t}\end{aligned}$$

so that we can also write the shear strain as

$$E_{nt} = \frac{1}{2} \left( \mathbf{t} \cdot \frac{d\mathbf{u}}{ds_n} + \mathbf{n} \cdot \frac{d\mathbf{u}}{ds_t} + \frac{d\mathbf{u}}{ds_n} \cdot \frac{d\mathbf{u}}{ds_t} \right)$$

If the strains are small enough so that we can neglect the products of the displacement gradients in these definitions, which is often the case, then we see that for small normal and shear strains we have

$$\begin{aligned}e_{nn} &= \mathbf{n} \cdot \frac{d\mathbf{u}}{ds_n} \\ e_{nt} &= \frac{1}{2} \left( \mathbf{n} \cdot \frac{d\mathbf{u}}{ds_t} + \mathbf{t} \cdot \frac{d\mathbf{u}}{ds_n} \right)\end{aligned}$$

Since these results are valid for any directions  $\mathbf{n}$  and  $\mathbf{t}$ , we can compute the strains (both finite and small) for line segments along the Cartesian  $(x, y, z)$  axes by simply making particular choices for  $\mathbf{n}$  and  $\mathbf{t}$ . Consider, for example, letting  $\mathbf{n}$  lie along the  $x$ -axis and  $\mathbf{t}$  along the  $y$  axis. Then  $\mathbf{n} = \mathbf{e}_1$ ,  $\mathbf{t} = \mathbf{e}_2$  and we have

$$E_{xx} = E_{11} = \mathbf{e}_1 \cdot \frac{\partial \mathbf{u}}{\partial x_1} + \frac{1}{2} \frac{\partial \mathbf{u}}{\partial x_1} \cdot \frac{\partial \mathbf{u}}{\partial x_1}$$

$$e_{xx} = e_{11} = \mathbf{e}_1 \cdot \frac{\partial \mathbf{u}}{\partial x_1}$$

$$E_{xy} = E_{12} = \frac{1}{2} \left( \mathbf{e}_2 \cdot \frac{\partial \mathbf{u}}{\partial x_2} + \mathbf{e}_1 \cdot \frac{\partial \mathbf{u}}{\partial x_1} + \frac{\partial \mathbf{u}}{\partial x_1} \cdot \frac{\partial \mathbf{u}}{\partial x_2} \right)$$

$$e_{xy} = e_{12} = \frac{1}{2} \left( \mathbf{e}_2 \cdot \frac{\partial \mathbf{u}}{\partial x_2} + \mathbf{e}_1 \cdot \frac{\partial \mathbf{u}}{\partial x_1} \right)$$

In exactly the same fashion we can obtain the other normal stresses and shear stresses. Note that we can write both the normal strains and shear strains in exactly the same form since we have

$$E_{ij} = \frac{1}{2} \left( \mathbf{e}_i \cdot \frac{\partial \mathbf{u}}{\partial x_j} + \mathbf{e}_j \cdot \frac{\partial \mathbf{u}}{\partial x_i} + \frac{\partial \mathbf{u}}{\partial x_i} \cdot \frac{\partial \mathbf{u}}{\partial x_j} \right) \quad (i, j = 1, 2, 3)$$

$$e_{ij} = \frac{1}{2} \left( \mathbf{e}_i \cdot \frac{\partial \mathbf{u}}{\partial x_j} + \mathbf{e}_j \cdot \frac{\partial \mathbf{u}}{\partial x_i} \right) \quad (i, j = 1, 2, 3)$$

These results are still in vector notation. However, if we write the displacement vector  $\mathbf{u}$  in terms of its scalar components along the  $(x, y, z)$  axes, i.e. we let

$$\mathbf{u} = \sum_{i=1}^3 u_i \mathbf{e}_i$$

$$\frac{\partial \mathbf{u}}{\partial x_j} = \sum_{i=1}^3 \frac{\partial u_i}{\partial x_j} \mathbf{e}_i$$

then the strains can be written directly in terms of these components as

$$E_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{k=1}^3 \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right)$$

$$e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

In a similar fashion we can write the strains  $E_{mn}$  and  $E_m$  in purely scalar forms in terms of the displacement components. Consider first  $E_{mn}$ . Since

$$\mathbf{n} = \sum_{i=1}^3 n_i \mathbf{e}_i = \frac{d\mathbf{r}_n}{ds_n} = \sum_{i=1}^3 \frac{dx_i}{ds_n} \mathbf{e}_i$$

$$\frac{d\mathbf{u}}{ds_n} = \sum_{i=1}^3 \frac{du_i}{ds_n} \mathbf{e}_i = \sum_{i=1}^3 \left( \sum_{j=1}^3 \frac{\partial u_i}{\partial x_j} \frac{dx_j}{ds_n} \right) \mathbf{e}_i = \sum_{i=1}^3 \sum_{j=1}^3 \frac{\partial u_i}{\partial x_j} n_j \mathbf{e}_i$$

we find

$$E_{mn} = \sum_{i=1}^3 \sum_{j=1}^3 \left( \frac{\partial u_i}{\partial x_j} + \frac{1}{2} \sum_{k=1}^3 \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right) n_i n_j$$

Note that we can write

$$\begin{aligned} \frac{\partial u_i}{\partial x_j} &= \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \\ &= e_{ij} + \omega_{ij} \end{aligned}$$

where  $e_{ij}$  are the small strain components we defined earlier and  $\omega_{ij}$  can be shown to be the location average rotation in the body. But the matrix of  $\omega_{ij}$  terms is antisymmetric, i.e.  $\omega_{ij} = -\omega_{ji}$  so that

$$\sum_{i=1}^3 \sum_{j=1}^3 \omega_{ij} n_i n_j = 0$$

This means that we can rewrite the expression for  $E_{mn}$  in the form

$$\begin{aligned} E_{mn} &= \sum_{i=1}^3 \sum_{j=1}^3 \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{k=1}^3 \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right) n_i n_j \\ &= \sum_{i=1}^3 \sum_{j=1}^3 E_{ij} n_i n_j \end{aligned}$$

or, equivalently

$$E_{mn} = \{n\}^T [E] \{n\}$$

In an entirely similar fashion we find the shear strain  $E_{nt}$  becomes

$$\begin{aligned} E_{nt} &= \sum_{i=1}^3 \sum_{j=1}^3 \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{k=1}^3 \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right) n_i t_j \\ &= \sum_{i=1}^3 \sum_{j=1}^3 E_{ij} n_i t_j \end{aligned}$$

or

$$E_{nt} = \{n\}^T [E] \{t\}$$

When the strains are small, we can simply neglect the product of displacement derivatives and find, analogously

$$\begin{aligned} e_{mn} &= \sum_{i=1}^3 \sum_{j=1}^3 \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) n_i n_j \\ &= \sum_{i=1}^3 \sum_{j=1}^3 e_{ij} n_i n_j \\ e_{nt} &= \sum_{i=1}^3 \sum_{j=1}^3 \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) n_i t_j \\ &= \sum_{i=1}^3 \sum_{j=1}^3 e_{ij} n_i t_j \end{aligned}$$

or, in matrix notation again

$$\begin{aligned} e_{mn} &= \{n\}^T [e] \{n\} \\ e_{nt} &= \{n\}^T [e] \{t\} \end{aligned}$$

If we compare these results to our previous results for stresses, we see that they are identical in form if we simply make the replacements  $\sigma_{mn}, \sigma_{nt} \rightarrow E_{mn}, E_{nt}$ ,  $\sigma_{ij} \rightarrow E_{ij}$  (or  $\sigma_{mn}, \sigma_{nt} \rightarrow e_{mn}, e_{nt}$ ,  $\sigma_{ij} \rightarrow e_{ij}$ ). Thus all our previous equations involving stress transformations, principal stresses, Mohr's circles, etc. remain valid in terms of strains as well. For example, using these replacements we can write the strain transformation equations relating the small strains  $e'_{ij}$  (along a set of  $(x'_1, x'_2, x'_3)$  axes) to the small strains  $e_{ij}$  (along a set of  $(x_1, x_2, x_3)$  axes) through the direction cosines as

$$e'_{mn} = \sum_{i=1}^3 \sum_{j=1}^3 l_{im} l_{jn} e_{ij}$$

or

$$[e'] = [l]^T [e] [l]$$

## Dilatation

Since the normal strains are measures of changes of length, they are also related to the change of volume occurring in the body. We have from the definitions of the strains

$$\Delta x' \cong \Delta x + e_{xx} \Delta x$$

$$\Delta y' \cong \Delta y + e_{yy} \Delta y$$

$$\Delta z' \cong \Delta z + e_{zz} \Delta z$$

so that to first order (neglecting products of small strains)

$$\Delta x' \Delta y' \Delta z' = \Delta x \Delta y \Delta z + (e_{xx} + e_{yy} + e_{zz}) \Delta x \Delta y \Delta z$$

For a small element volume  $V = \Delta x \Delta y \Delta z$ , the relative change of volume of that element is defined as the dilatation,  $\Delta$ , given by

$$\begin{aligned} \Delta &\equiv \frac{\Delta V}{V} = e_{xx} + e_{yy} + e_{zz} \\ &= \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \end{aligned}$$

## Summary – small strains

Since all the problems we will solve will make the small strain assumption, below we have summarized the results obtained previously for small strains in explicit component forms.

$$e_{xx} = \frac{\partial u_x}{\partial x}, \quad e_{yy} = \frac{\partial u_y}{\partial y}, \quad e_{zz} = \frac{\partial u_z}{\partial z}$$

$$e_{xy} = \frac{\gamma_{xy}}{2} = \frac{1}{2} \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)$$

$$e_{xz} = \frac{\gamma_{xz}}{2} = \frac{1}{2} \left( \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right)$$

$$e_{yz} = \frac{\gamma_{yz}}{2} = \frac{1}{2} \left( \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right)$$

$$e_{nn} = e_{xx}n_x^2 + e_{yy}n_y^2 + e_{zz}n_z^2 \\ + 2e_{xy}n_xn_y + 2e_{xz}n_xn_z + 2e_{yz}n_yn_z$$

$$e_{nt} = e_{xx}n_x t_x + e_{yy}n_y t_y + e_{zz}n_z t_z \\ + e_{xy}(n_x t_y + n_y t_x) \\ + e_{xz}(n_x t_z + n_z t_x) \\ + e_{yz}(n_y t_z + n_z t_y)$$

$$\begin{bmatrix} e'_{xx} & e'_{xy} & e'_{xz} \\ e'_{yx} & e'_{yy} & e'_{yz} \\ e'_{zx} & e'_{zy} & e'_{zz} \end{bmatrix} = \begin{bmatrix} l_{xx} & l_{yx} & l_{zx} \\ l_{xy} & l_{yy} & l_{zy} \\ l_{xz} & l_{yz} & l_{zz} \end{bmatrix} \begin{bmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{bmatrix} \begin{bmatrix} l_{xx} & l_{xy} & l_{xz} \\ l_{yx} & l_{yy} & l_{yz} \\ l_{zx} & l_{zy} & l_{zz} \end{bmatrix}$$